Combinatorics of 4-dimensional Resultant Polytopes

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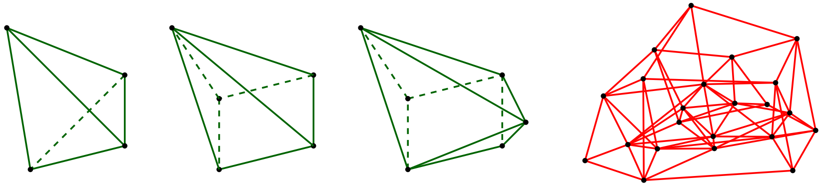
Resultant polytopes

- Algebra: generalization of the resultant polynomial degree
- Geometry: Minkowski summands of secondary polytopes
- Applications: support computation $\rightarrow$ discriminant and resultant computation



Polytopes and Algebra

- Given $n+1$ polynomials on n variables.
- Supports (set of exponents of monomials with non-zero coefficient) $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$.
- The resultant R is the polynomial in the coefficients of a system of polynomials which vanishes if there exists a common root in the torus of the given polynomials.
- The resultant polytope $N(R)$, is the convex hull of the support of R, i.e. the Newton polytope of the resultant.

$$f_0(x) = ax^2 + b$$
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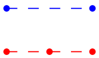
A_1



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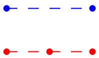

$$R(a,b,c,d,e) = ad^2b + c^2b^2 - 2caeb + a^2e^2$$

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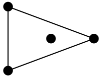


Polytopes and Algebra

The case of linear polynomials

$$f_0(x, y) = ax + by + c$$

A_0 

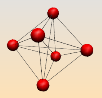
$$f_1(x, y) = dx + ey + f$$

A_1 

$$f_2(x, y) = gx + hy + i$$

A_2 

$$R(a, b, c, d, e, f, g, h, i) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$N(R)$ 

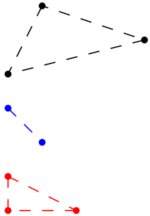
4-dimensional Birkhoff polytope

Polytopes and Algebra

$f_0(x,y) = axy^2 + x^4y + c$ A_0

$f_1(x,y) = dx + ey$ A_1

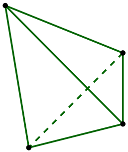
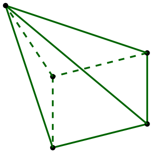
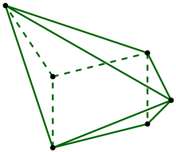
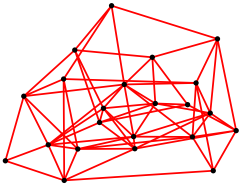
$f_2(x,y) = gx^2 + hy + i$ A_2



Q: How $N(R)$ looks like in the general case ?

Resultant polytopes: Motivation

- Algebra: useful to express the solvability of polynomial systems, generalizes the notion of the degree of the resultant
- Geometry: Minkowski summands of secondary polytopes, equival. classes of secondary vertices, generalize Birkhoff polytopes
- Applications: support computation $\rightarrow$ discriminant and resultant computation, implicitization of parametric hypersurfaces

Existing work

- [GKZ'90] Univariate case / general dimensional $N(R)$

- [Sturmfels'94] Multivariate case / up to 3 dimensional $N(R)$

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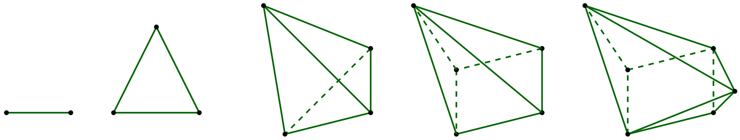

One step beyond... 4-dimensional N(R)

- Polytope $P \subseteq \mathbb{R}^4$; f-vector is the vector of its face cardinalities.
- Call vertices, edges, ridges, facets, the 0,1,2,3-d, resp., faces of P.
- f-vectors of 4-dimensional N(R)

(5, 10, 10, 5)
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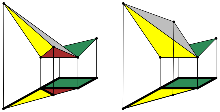
Computation of resultant polytopes



- respol software [Emiris-F-Konaxis-Peñaranda '12]
  - lower bounds
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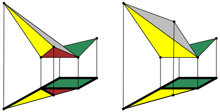

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Main result

Theorem

Given $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$ with $N(R)$ of dimension 4. Then $N(R)$ are degenerations of the polytopes in following cases.

(i) *All $|A_i| = 2$, except for one with cardinality 5, is a 4-simplex with f-vector $(5, 10, 10, 5)$.*

(ii) *All $|A_i| = 2$, except for two with cardinalities 3 and 4, has f-vector $(10, 26, 25, 9)$.*

(iii) *All $|A_i| = 2$, except for three with cardinality 3, maximal number of ridges is $\tilde{f}_2 = 66$ and of facets $\tilde{f}_3 = 22$. Moreover, $22 \leq \tilde{f}_0 \leq 28$, and $66 \leq \tilde{f}_1 \leq 72$. The lower bounds are tight.*

- Degenarations can only decrease the number of faces.
- Focus on new case (iii), which reduces to $n = 2$ and each $|A_i| = 3$.
- Previous upper bound for vertices yields 6608 [Sturmfels'94].

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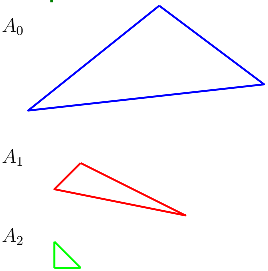
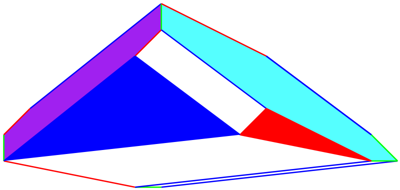
A subdivision S of $A_0 + A_1 + \cdots + A_n$ is mixed when its cells have expressions as Minkowski sums of convex hulls of point subsets in A_i's.

Example





mixed subdivision S of $A_0 + A_1 + A_2$

Proposition (Sturmfels'94)

A regular mixed subdivision S of $A_0 + A_1 + \cdots + A_n$ corresponds to a face of $\mathrm{N}(\mathrm{R})$ which is the Minkowski sum of the resultant polytopes of the cells (subsystems) of S.

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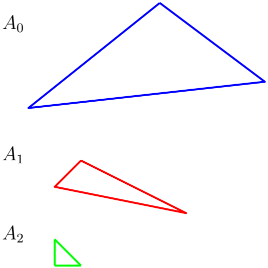

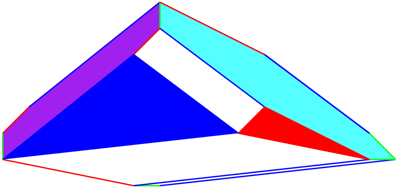

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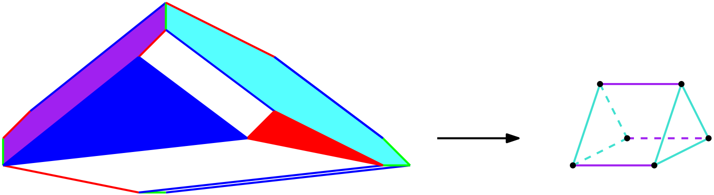
- white, blue, red cells $\longrightarrow$ N(R) vertex
- purple cell $\longrightarrow$ N(R) segment
- turquoise cell $\longrightarrow$ N(R) triangle



subd. S of $A_0 + A_1 + A_2$ Mink. sum of $N(R)$ triangle and $N(R)$ segment

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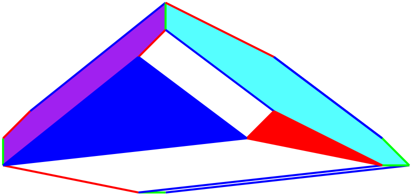

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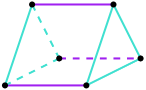

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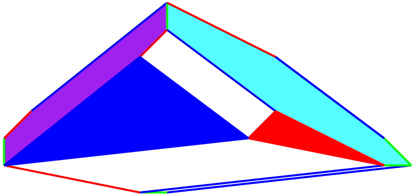

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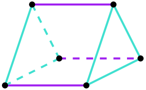

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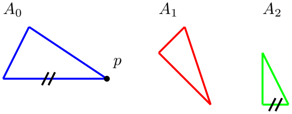
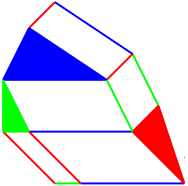
Tool (2): Input genericity

Proposition

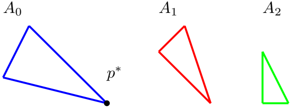
Input genericity maximizes the number of resultant polytope faces.

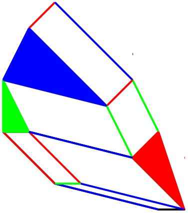
Proof idea





$N(R)$ f-vector: $(14, 38, 36, 12)$





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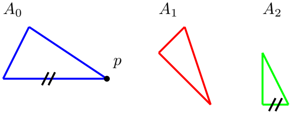

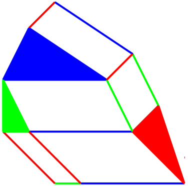

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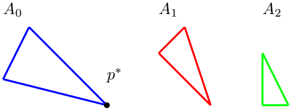

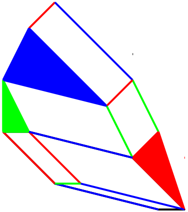

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$\rightarrow$ For upper bounds on the number of $N(R)$ faces consider generic inputs, i.e. no parallel edges.

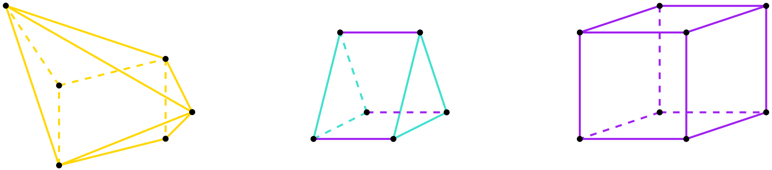
Facets of 4-d resultant polytopes

Lemma

All the possible types of $\mathrm{N}(\mathrm{R})$ facets are

- *resultant facet: 3-d N(R)*
- *prism facet: 2-d N(R) (triangle)* $+$ *1-d N(R)*
- *cube facet: 1-d N(R)* $+$ *1-d N(R)* $+$ *1-d N(R)*

$3D$



Facets of 4-d resultant polytopes

Lemma

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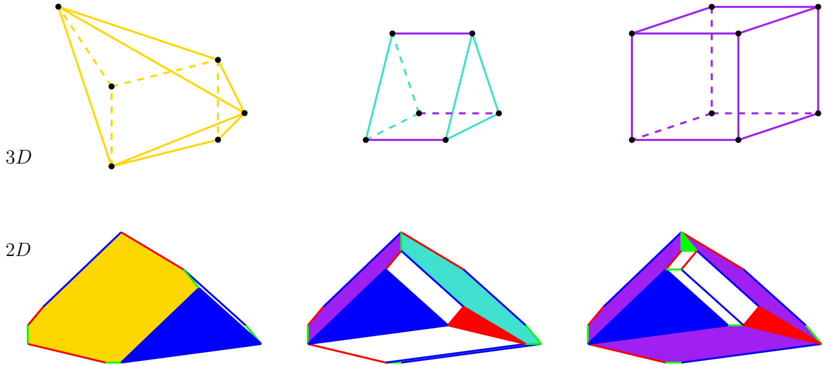


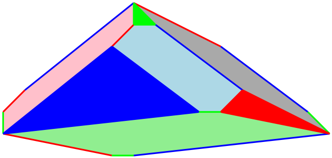
Counting facets

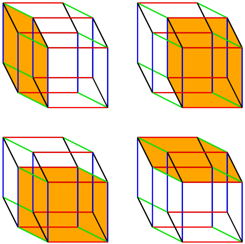
Lemma

There can be at most $9, 9, 4$ resultant, prism, cube facets, resp., and this is tight.

Proof idea

- Unique subdivision that corresponds to 4 cube facets





Faces of 4-d resultant polytopes

Lemma

The maximal number of ridges of $N(R)$ *is* $\tilde{f_2} = 66$. *Moreover,* $\tilde{f_1} = \tilde{f_0} + 44$, $22 \leq \tilde{f_0} \leq 28$, *and* $66 \leq \tilde{f_1} \leq 72$. *The lower bounds are tight.*

Elements of proof

- [Kalai87]

$$f_1 + \sum_{i \geq 4} (i-3) f_2^i \geq d f_0 - \binom{d+1}{2},$$

where f_2^i is the number of 2-faces which are i-gons.

Open problems & a conjecture

Open

The maximum f-vector of a 4d-resultant polytope is $(22, 66, 66, 22)$.

Open

Explain symmetry of f-vectors of 4d-resultant polytopes.

Conjecture

$$f_0(d) \leq 3 \cdot \sum_{\|S\|=d-1} \prod_{i \in S} \tilde{f_0}(i)$$

where S is any multiset with elements in $\{1, \ldots, d-1\}$, $\|S\| := \sum_{i \in S} i$, and $\tilde{f_0}(i)$ is the maximum number of vertices of a i-dimensional $N(R)$.

- The only bound in terms of d is $(3d-3)^{2d^2}$ [Sturmfels'94], yielding $\tilde{f_0}(5) \leq 12^{50}$ whereas our conjecture yields $\tilde{f_0}(5) \leq 231$.

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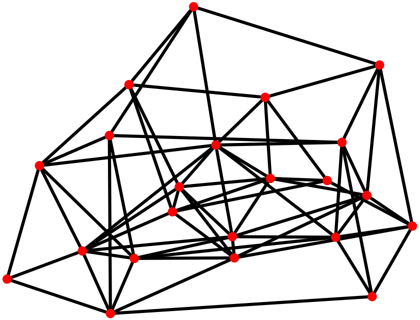

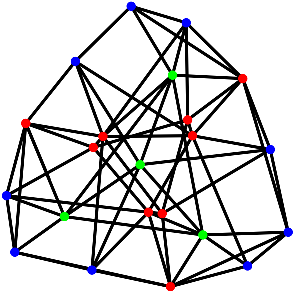

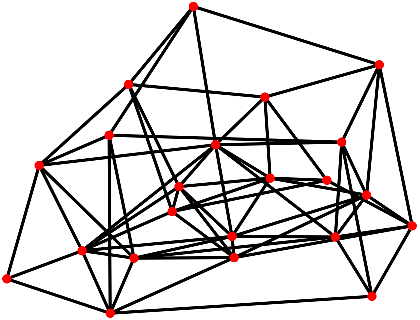

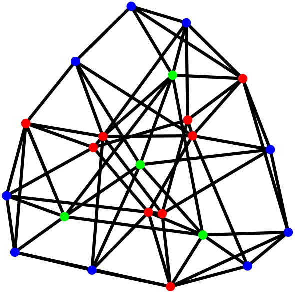

Thank you!