Faster Geometric Algorithms via Dynamic Determinant Computation

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joint work with L. Peñaranda (now IMPA, Rio de Janeiro)

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Geometric algorithms and predicates

Setting

- geometric algorithms $\longrightarrow$ sequence of geometric predicates
- many geometric predicates $\longrightarrow$ determinants
- Hi-dim: as dimension grows predicates become more expensive

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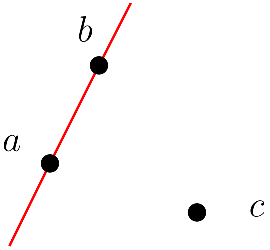
Examples

- Orientation: Does c lie on, left or right of ab?

$$\begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix} \gtreqless 0$$



Geometric algorithms and predicates

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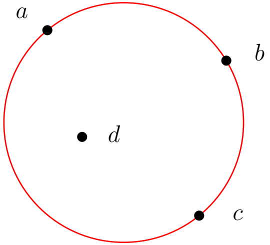
- geometric algorithms $\longrightarrow$ sequence of geometric predicates
- many geometric predicates $\longrightarrow$ determinants
- Hi-dim: as dimension grows predicates become more expensive

Examples

- inCircle: Does d lie on, inside or outside of abc?

$$\begin{vmatrix} a_x & a_y & a_x^2 + a_y^2 & 1 \\ b_x & b_y & b_x^2 + b_y^2 & 1 \\ c_x & c_y & c_x^2 + c_y^2 & 1 \\ d_x & d_y & d_x^2 + d_y^2 & 1 \end{vmatrix} \gtreqless 0$$



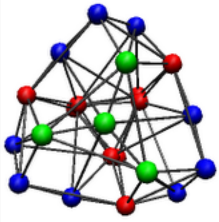
Outline

1. Motivation and existing work
2. Dynamic determinant computation
3. Geometric algorithms: convex hull
4. Implementation - experiments

Outline

Motivation

- Algorithms for resultant polytopes [Emiris,F,Konaxis,Peñaranda SoCG'12] [YuJensen'12] (discriminant polytopes [Emiris, F, Dickenstein])
  `Respol`: compute resultant, discriminant polytopes up to dim. 6



4-d resultant polytope

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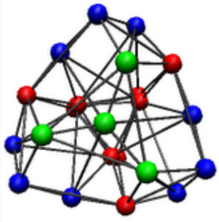
- Counting lattice points in polyhedra [Barvinock'99] [DeLoera et.al'04], Integer convex optimization [Grötschel,Lovász,Schrijver'88]



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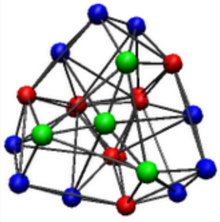

4-d resultant polytope

Focusing on ...

- Hi-dim CompGeom ($3 < d < 10$)
- Computation over the integers

Existing work

Determinant: exact computation

Given matrix $A \subseteq \mathbb{R}^{d \times d}$

- Theory: State-of-the-art $O(d^{\omega})$, $\omega \sim 2.37$ [CoppersmithWinograd90]
- Practice: Gaussian elimination, $O(d^3)$

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Division-free determinant algorithms

- Laplace (cofactor) expansion, $O(d!)$
- [Rote01], $O(d^4)$
- [Bird11], $O(d^{\omega+1})$

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Dynamic determinant computation

Dynamic transitive closure in graphs [Sankowski FOCS'04]

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Dynamic Determinant Computations

One-column update problem

Given matrix $A \subseteq \mathbb{R}^{d \times d}$, answer queries for $\det(A)$ when i-th column of A, $(A)_i$, is replaced by $u \subseteq \mathbb{R}^d$.

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Solution: Sherman-Morrison formula (1950)

$$A'^{-1} = A^{-1} - \frac{(A^{-1}(u - (A)_i))\ (e_i^T A^{-1})}{1 + e_i^T A^{-1}(u - (A)_i)}$$

$$\det(A') = (1 + e_i^T A^{-1}(u - (A)_i)\det(A)$$

- Only vector$\times$vector, vector$\times$matrix $\longrightarrow$ Complexity: $O(d^2)$
- Algorithm: dyn_inv

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Variant Sherman-Morrison formulas

- Q: What happens if we work over the integers?

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- Q: What happens if we work over the integers?
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- Complexity: $O(d^2)$ Algorithm: dyn_adj

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The convex hull problem

Definition

The *convex hull* of $\mathcal{A} \subseteq \mathbb{R}^d$ is the smallest convex set containing $\mathcal{A}$.

Example

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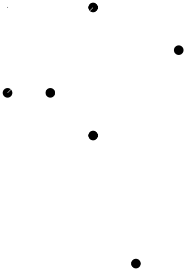
Incremental convex hull - Beneath-and-Beyond



- connect each outer point to the visible segments
- visibility test = orientation test

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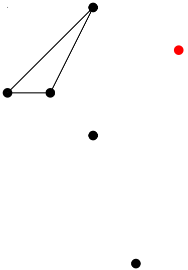

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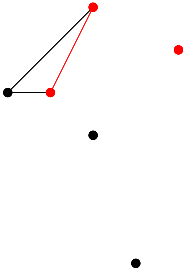

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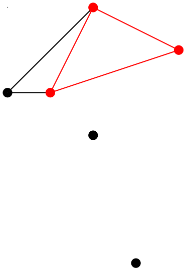

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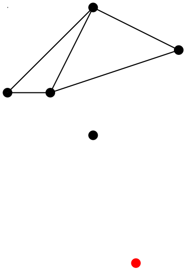

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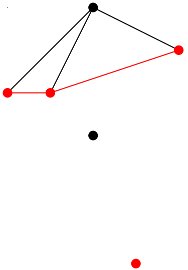

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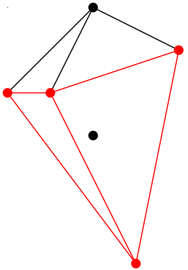

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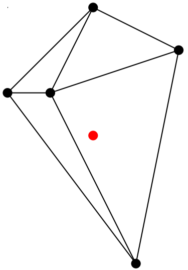

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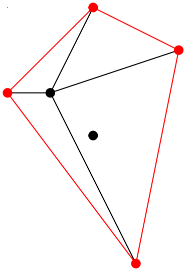

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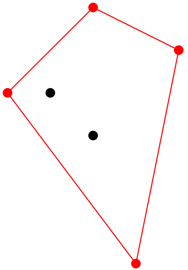

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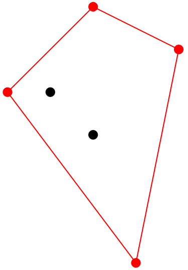

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Similar problems: Delaunay, regular triangulations, point-location

The convex hull problem: REVISITED

- convex hull $\longrightarrow$ sequence of similar orientation predicates
- take advantage of computations done in previous steps

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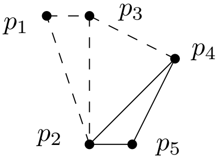
Incremental convex hull construction $\longrightarrow$ 1-column updates

$$A = \begin{array}{|c|c|c|} \hline p_2 & p_4 & p_5 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$



- $\text{Orientation}(p_2, p_4, p_5) = \det(A)$

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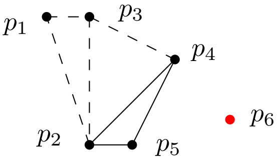
Incremental convex hull construction $\longrightarrow$ 1-column updates

$$A' = \begin{array}{|c|c|c|} \hline \boxed{p_6} & p_4 & p_5 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$



- Orientation$(p_6, p_4, p_5) = \det(A')$ in $O(d^2)$

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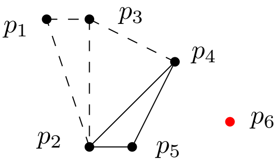

- Orientation$(p_6, p_4, p_5) = \det(A')$ in $O(d^2)$
- Store $\det(A)$, A^{-1} + update $\det(A')$, A'^{-1} (Sherman-Morrison)

Dynamic determinants in geometric algorithms

Given $\mathcal{A} \subseteq \mathbb{R}^d$, $n = |\mathcal{A}|$ and $t =$ the number of cells

Theorem

Orientation predicates in increm. convex hull: $O(d^2)$ *(space:* $O(d^2 t)$*)*
Proof: *update det(*A'*),* A'^{-1}

Corollary

Orientation predicates involved in point-location: $O(d)$ *(space:* $O(d^2 t)$*)*
Proof: *query point never enters data-set* $\rightarrow$ *update only det(*A'*)*

Corollary

Incremental convex hull and volume comput.: $O(d^{\omega+1} n t) \rightarrow O(d^3 n t)$

Outline

Determinant implementation

Dynamic determinant computation

- C++, GNU Multiple Precision arithmetic library (GMP)
- Implement dyn_inv & dyn_adj

Exact determinant computation software

- LU decomposition (CGAL)
- optimized LU decomposition (Eigen)
- asymptotically optimal algorithms (LinBox)
- Maple's default determinant algorithm (Maple 14)
- Bird's algorithm (our implementation)
- Laplace (cofactor) expansion (our implementation)

Determinant experiments

1-column updates in $A \subseteq \mathbb{Q}^{d \times d}$ (uniform distr. rational coefficients)

d	Bird	CGAL	Eigen	Laplace	Maple	dyn_inv	dyn_adj
3	.013	.021	.014	.008	.058	.046	.023
4	.046	.050	.033	.020	.105	.108	.042
5	.122	.110	.072	.056	.288	.213	.067
6	.268	.225	.137	.141	.597	.376	.102
7	.522	.412	.243	.993	.824	.613	.148
8	.930	.710	.390	–	1.176	.920	.210
9	1.520	1.140	.630	–	1.732	1.330	.310
10	2.380	1.740	.940	–	2.380	1.830	.430

spec: Intel Core i5-2400 3.1GHz, 6MB L2 cache, 8GB RAM, 64-bit Debian GNU/Linux

Determinant experiments

1-column updates in $A \subseteq \mathbb{Q}^{d \times d}$ (uniform distr. integer coefficients)

d	Bird	CGAL	Eigen	Laplace	Linbox	Maple	dyn_inv	dyn_adj
3	.002	.021	.013	.002	.872	.045	.030	.008
4	.012	.041	.028	.005	1.010	.094	.058	.015
5	.032	.080	.048	.016	1.103	.214	.119	.023
6	.072	.155	.092	.040	1.232	.602	.197	.033
7	.138	.253	.149	.277	1.435	.716	.322	.046
8	.244	.439	.247	–	1.626	.791	.486	.068
9	.408	.689	.376	–	1.862	.906	.700	.085
10	.646	1.031	.568	–	2.160	1.014	.982	.107
11	.956	1.485	.800	–	10.127	1.113	1.291	.133
12	1.379	2.091	1.139	–	13.101	1.280	1.731	.160
13	1.957	2.779	1.485	–	–	1.399	2.078	.184

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Convex hull implementation

Hashed dynamic determinants

- dyn_inv & dyn_adj + hash table (dets & matrices)
- plug into geometric software $\longrightarrow$ geometric predicates

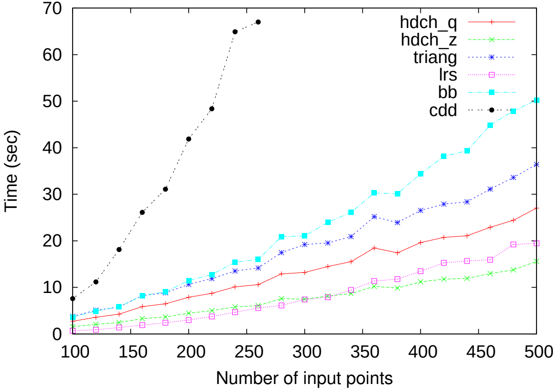
Convex hull software

- randomized incremental (triang/CGAL)
- beneath-and-beyond (bb) (polymake)
- double description method (cdd)
- gift-wrapping with reverse search (lrs)

triang/CGAL + hashed dynamic determinants = `hdch_z` or `hdch_q`

Convex hull experiments

6-dim points, integer coeffs uniformly distributed inside a 6-ball



- hdch_q (hdch_z) rational (integer) arithmetic
- drawback: memory consumption

500 6D points	
triang	0.1GB
hdch_z	2.8GB
hdch_q	3.8GB

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Point location experiments

- triangulation of $\mathcal{A} \subseteq \mathbb{Z}^{\mathrm{d}}$ points uniformly distibuted on a d-ball surface
- 1K and 1000K query points uniformly distibuted on a d-cube

	d	$\lvert\mathcal{A}\rvert$	preprocess time (sec)	ds mem (MB)	# cells triang	query time (sec) 1K	query time (sec) 1000K
`hdch_z`	8	120	45.20	6913	319438	0.41	392.55
`triang`	8	120	156.55	134	319438	14.42	14012.60
`hdch_z`	9	70	45.69	6826	265874	0.28	276.90
`triang`	9	70	176.62	143	265874	13.80	13520.43
`hdch_z`	10	50	43.45	6355	207190	0.27	217.45
`triang`	10	50	188.68	127	207190	14.40	14453.46
`hdch_z`	11	39	38.82	5964	148846	0.18	189.56
`triang`	11	39	181.35	122	148846	14.41	14828.67

- up to 78 times faster using up to 50 times more memory

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Conclusions and Future work

- Orientation predicates. CH: $O(d^2)$, point location: $O(d)$
- Efficient (division free) dynamic determinant implementation
- More efficient CH implementation, $78\times$ speed-up in point location
- The code: `http://hdch.sourceforge.net`

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- filtered computations
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