Regular triangulations and resultant polytopes

Vissarion Fisikopoulos
joint work with Ioannis Z. Emiris and Christos Konaxis

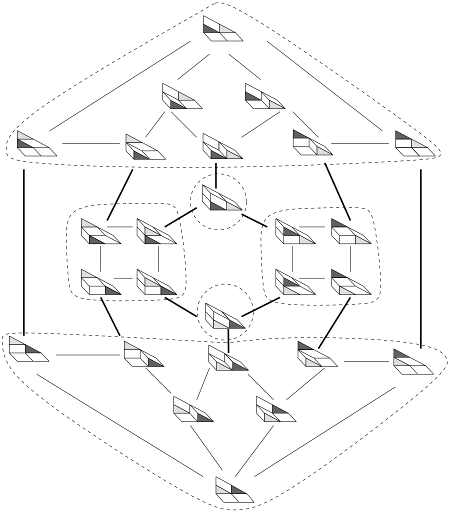
National and Kapodistrian University of Athens
Department of Informatics and Telecommunications

22 March 2010

Motivation

- **Computation of Resultants**
  - solve polynomial systems
- **Implicitization**
  - parametric (hyper)surfaces

- Reduction to graph enumeration problems



Outline

1. Triangulations, mixed subdivisions, and polynomial systems
   - triangulations - mixed subdivisions (Cayley trick)
   - mixed subdivisions - Newton polytope of the Resultant

2. Mixed cell configurations and cubical flips
   - define equivalence classes of mixed subdivisions
   - flips between classes of mixed subdivisions

Outline

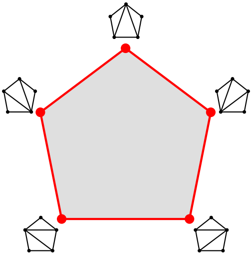
1. Triangulations, mixed subdivisions, and polynomial systems
   - triangulations - mixed subdivisions (Cayley trick)
   - mixed subdivisions - Newton polytope of the Resultant
2. Mixed cell configurations and cubical flips
   - define equivalence classes of mixed subdivisions
   - flips between classes of mixed subdivisions

The Secondary Polytope

Let A a set of n points in $\mathbb{R}^d$.

Theorem [Gelfand-Kapranov-Zelevinsky]

To every point set A corresponds a Secondary polytope $\Sigma(A)$ with dimension $n - d - 1$. The vertices correspond to the regular triangulations of A and the edges to (bistellar) flips.



Enumeration of regular triangulations: [Rambau02], [Masada et al.96]

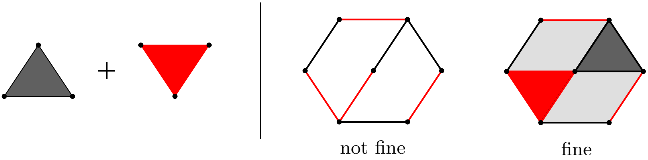
Mixed Subdivisions

Let $A_0, A_1, \ldots, A_k$ point sets in $\mathbb{R}^d$ and $A = A_0 + A_1 + \ldots + A_k$ their Minkowski sum.

Definition

A regular polyhedral subdivision of A is called **regular fine mixed subdivision** if for every cell σ

- $\sigma = F_0 + \cdots + F_k$ for certain subsets $F_0 \subseteq A_0, \ldots, F_k \subseteq A_k$
- all F_i are affinely independent and σ does not contain any other cell



The Cayley Trick

Definition

The **Cayley embedding** of $A_0, \dots, A_k$ in $\mathbb{R}^d$ is the point set

$$\mathcal{C}(A_0, \dots, A_k) = A_0 \times \{e_0\} \cup \cdots \cup A_k \times \{e_k\} \subseteq \mathbb{R}^d \times \mathbb{R}^k$$

where $e_0, \dots, e_k$ are an affine basis of $\mathbb{R}^k$.

Proposition (the Cayley trick)

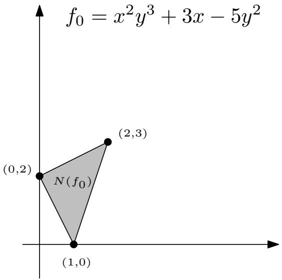
$$\left(\begin{array}{c} \text{regular} \\ \text{triangulations} \\ \text{of} \\ \mathcal{C}(A_0, \dots, A_k) \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \text{regular} \\ \text{fine mixed} \\ \text{subdivisions} \\ \text{of} \\ A_0 + \ldots + A_k \end{array} \right)$$

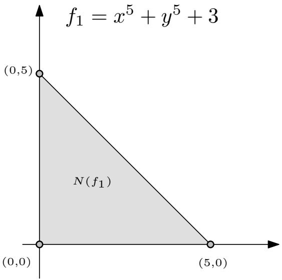
Resultant polytope

Let $f = f_0, f_1, \ldots, f_k$ where $f_i \in K[x_1, \ldots, x_k]$ with coefficients c_{ij}.

Definitions

- Given a polynomial f_i its **support** $sup(f_i)$ is the set of its exponent vectors and **Newton polytope** $N(f_i)$ is the convex hull of $sup(f_i)$.
- The **Resultant** of f is a polynomial $R \in K[c_{ij}]$ s.t. $R = 0$ iff f has a common root.
- The Newton polytope of the Resultant is the **Resultant polytope.**





i-mixed cells

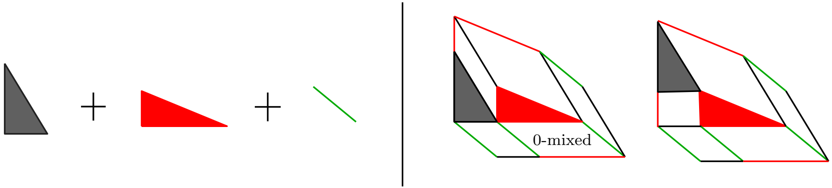
Let $A_0, A_1, \ldots, A_k$ s.t. $A_j = sup(f_j)$, $A = A_0 + A_1 + \cdots + A_k$

Definition

A cell σ of a mixed subdivision is called **i-mixed** if for all j exists $F_j \subseteq A_j$ s.t.

$$\sigma = F_0 + \cdots + F_{i-1} + F_i + F_{i+1} + \cdots + F_k$$

where $|F_j| = 2$ (edges) for all $j \neq i$ and $|F_i| = 1$ (vertex).



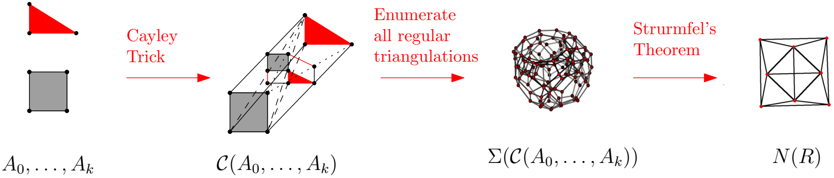
Resultant Extreme Terms

Theorem [Sturmfels]

Given a regular fine mixed subdivision of $A = A_0 + A_1 + \cdots + A_k$ *we get a unique vertex of the Resultant polytope* $N(R)$

- *there is a many to one mapping from regular fine mixed subdivisions to vertices of* $N(R)$
- *the construction depends only on* i*-mixed cells*

An Algorithm



Outline

1. Triangulations, mixed subdivisions, and polynomial systems
   - triangulations - mixed subdivisions (Cayley trick)
   - mixed subdivisions - Newton polytope of the Resultant

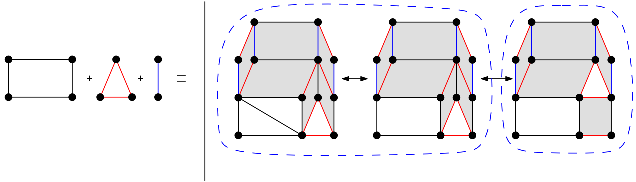
2. Mixed cell configurations and cubical flips
   - define equivalence classes of mixed subdivisions
   - flips between classes of mixed subdivisions

i - Mixed Cells Configurations

Generalizing mixed cells configurations of [MichielsVerschelde99]

Definition

i-mixed cells configurations are the equivalence classes of mixed subdivisions with the same i-mixed cells for all $i \in \{0, 1, \ldots, k\}$.



Proposition

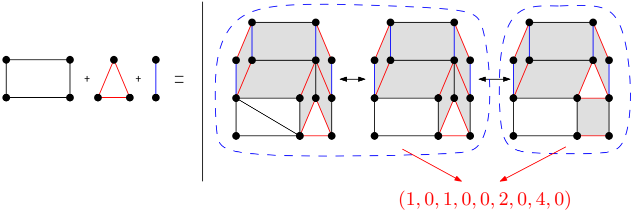
There exist flips that transform one i-mixed cell configuration to another by destroying at least one i-mixed cell.

i - Mixed Cells Configurations

Generalizing mixed cells configurations of [MichielsVerschelde99]

Definition

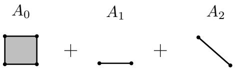
i-mixed cells configurations are the equivalence classes of mixed subdivisions with the same i-mixed cells for all $i \in \{0, 1, \ldots, k\}$.

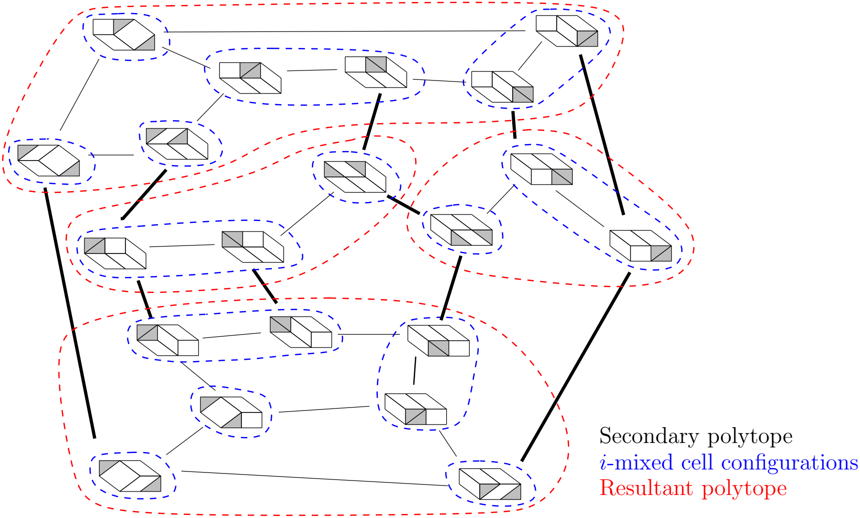


Proposition

There exist flips that transform one i-mixed cell configuration to another by destroying at least one i-mixed cell.

An example


A_0
A_1
A_2
+
+


Secondary polytope
i-mixed cell configurations
Resultant polytope

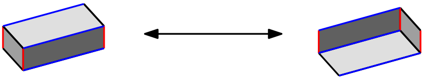
Cubical Flips

- Let S a regular fine mixed subdivision then S has a cubical flip iff exists a set $C \subseteq S$ of i-mixed cells s.t. $C = F_0 + F_1 + \cdots + F_k$ and

$$C = \begin{cases} C_0 = a_0 + F_1 + \cdots + F_i + \cdots + F_{k-1} + F_k \\ \vdots \\ C_i = F_0 + F_1 + \ldots + a_i + \cdots + F_{k-1} + F_k \\ \vdots \\ C_k = F_0 + F_1 + \cdots + F_i + \cdots + F_{k-1} + a_k \end{cases}$$

  where $a_i \in F_i \subseteq A_i, |a_i| = 1, |F_i| = 2.$

- The cubical flip supported on C of a subdivision S to a subdivision S' consist of changing every a_i with $F_i - \{a_i\}$.



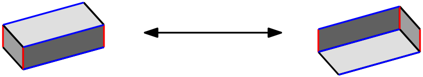
Cubical Flips

- Let S a regular fine mixed subdivision then S has a cubical flip iff exists a set $C \subseteq S$ of i-mixed cells s.t. $C = F_0 + F_1 + \cdots + F_k$ and

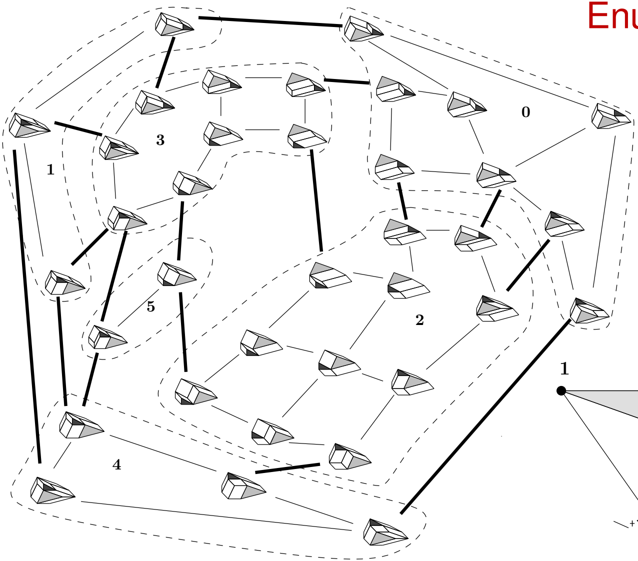
$$C' = \begin{cases} C'_0 = F_0 - \{a_0\} + F_1 + \cdots + F_i + \cdots + F_{k-1} + F_k \\ \vdots \\ C'_i = F_0 + F_1 + \ldots + F_i - \{a_i\} + \cdots + F_{k-1} + F_k \\ \vdots \\ C'_k = F_0 + F_1 + \cdots + F_i + \cdots + F_{k-1} + F_k - \{a_k\} \end{cases}$$

  where $a_i \in F_i \subseteq A_i, |a_i| = 1, |F_i| = 2$.

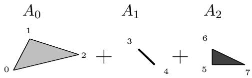
- The cubical flip supported on C of a subdivision S to a subdivision S' consist of changing every a_i with $F_i - \{a_i\}$.



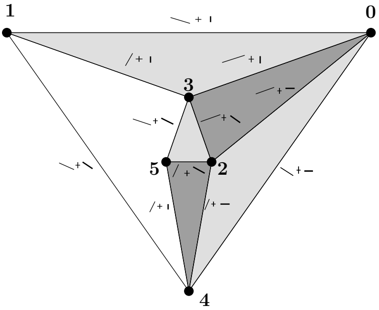
Enumerating $N(R)$



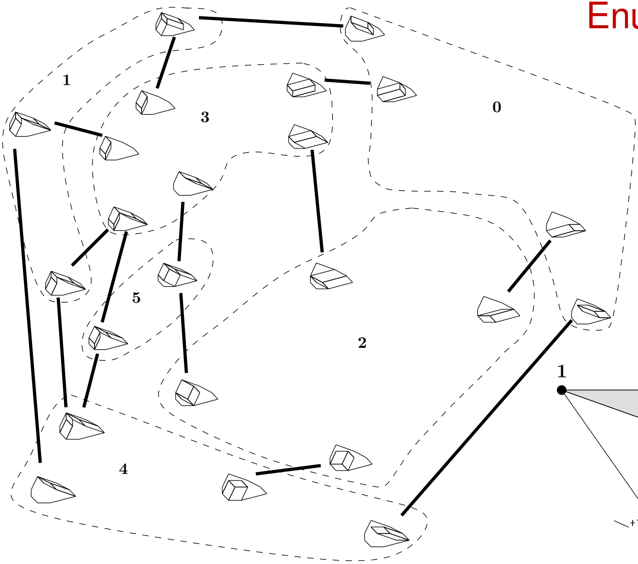
Secondary Polytope



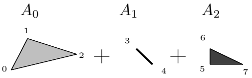
Resultant Polytope $N(R)$



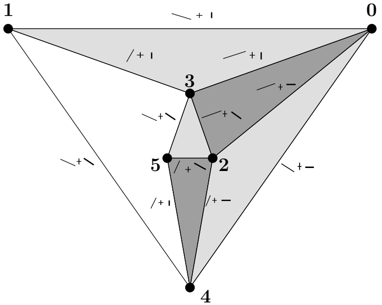
Enumerating $N(R)$



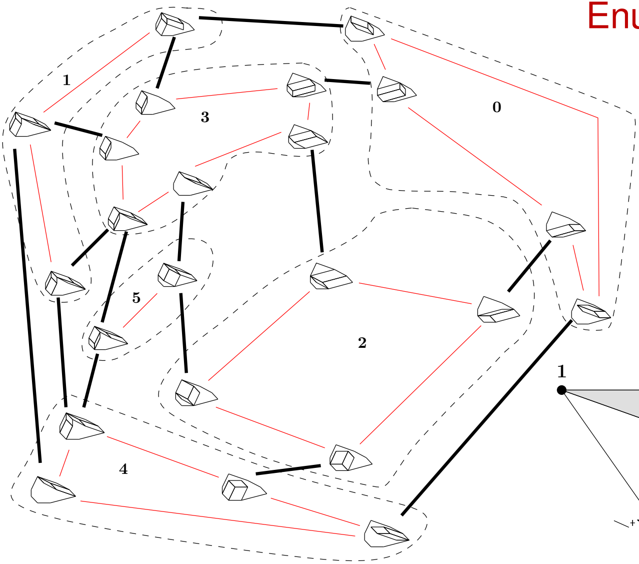
Secondary Polytope



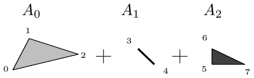
Resultant Polytope $N(R)$



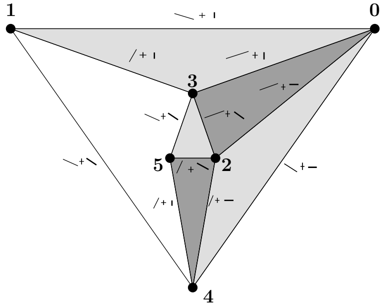
Enumerating $N(R)$



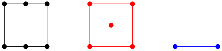
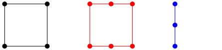
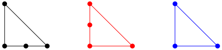
Secondary Polytope

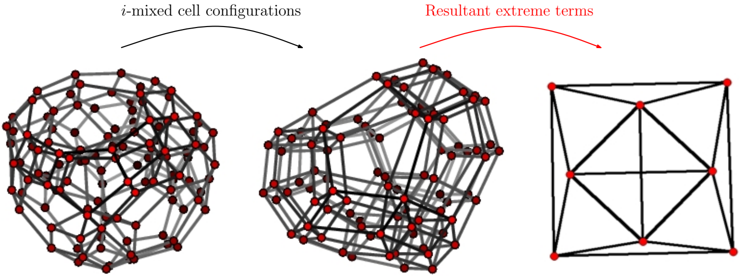


Resultant Polytope $N(R)$



Complexity

supports	# Secondary polytope vertices	i-mixed cell configurations	# Resultant polytope vertices
	122	98	8
	104148	43018	21
	76280	32076	95
	3540	3126	22



Secondary Polytope

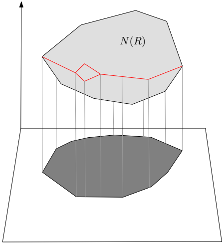
i-mixed cell configurations

Resultant Polytope

Conclusion - Future Work

- # Σ vertices $\geq$ # i-mixed cell configurations $\geq$ # $N(R)$ vertices
- Algorithmic test for cubical flips, disconnected graph of cubical flips
- Settle some easy cases
- Wiki page with experiments
  http://ergawiki.di.uoa.gr/index.php/Implicitization

- Enumerate Resultant polytope vertices
- $N(R)$ is a Minkowski summand of the Secondary polytope
  [MichielsCools00],[Sturmfels94]
- In some applications (e.g. implicitization) we need to compute only a silhouette w.r.t. a projection of $N(R)$ [EmirisKonaxisPalios07]



Thank You!