ΕρΓΑ Laboratory of Algebraic and Geometric Algorithms

Combinatorics of 4-dimensional resultant polytopes

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Polynomials and polytopes

- We consider $n+1$ **(Laurent) polynomials** with fixed support sets $A_i \subset \mathbf{Z}^n$, $i = 0, \dots, n$, in n unknowns $x = (x_1, x_2, \dots, x_n)$ over an algebraically closed field K, with symbolic coefficients $c_{i,a}$, $a \in A_i$.
- **Newton polytope** ($N(f)$) of a polynomial f is the convex hull of its support set.

The Cayley trick

- Given pointsets $A_0, \dots, A_n \subset \mathbb{Z}^n$, we define the pointset

$$\mathcal{A} := \bigcup_{i=0}^{n} (A_i \times \{e_i\}) \subset \mathbb{Z}^{2n}, \qquad (1)$$

where $e_0, \dots, e_n$ are an affine basis of $\mathbb{R}^n$: $e_0 = (0, \dots, 0)$, $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, $i = 1, \dots, n$.

- A subdivision S of $A_0 + A_1 + \cdots + A_n$ is **mixed** when its cells have expressions as Minkowski sums of convex hulls of point subsets in A_i's. **Regular tight mixed subdivisions** of $A_0 + \cdots + A_n$ are in bijection with **regular triangulations** of $\mathcal{A}$.

Secondary and resultant polytopes

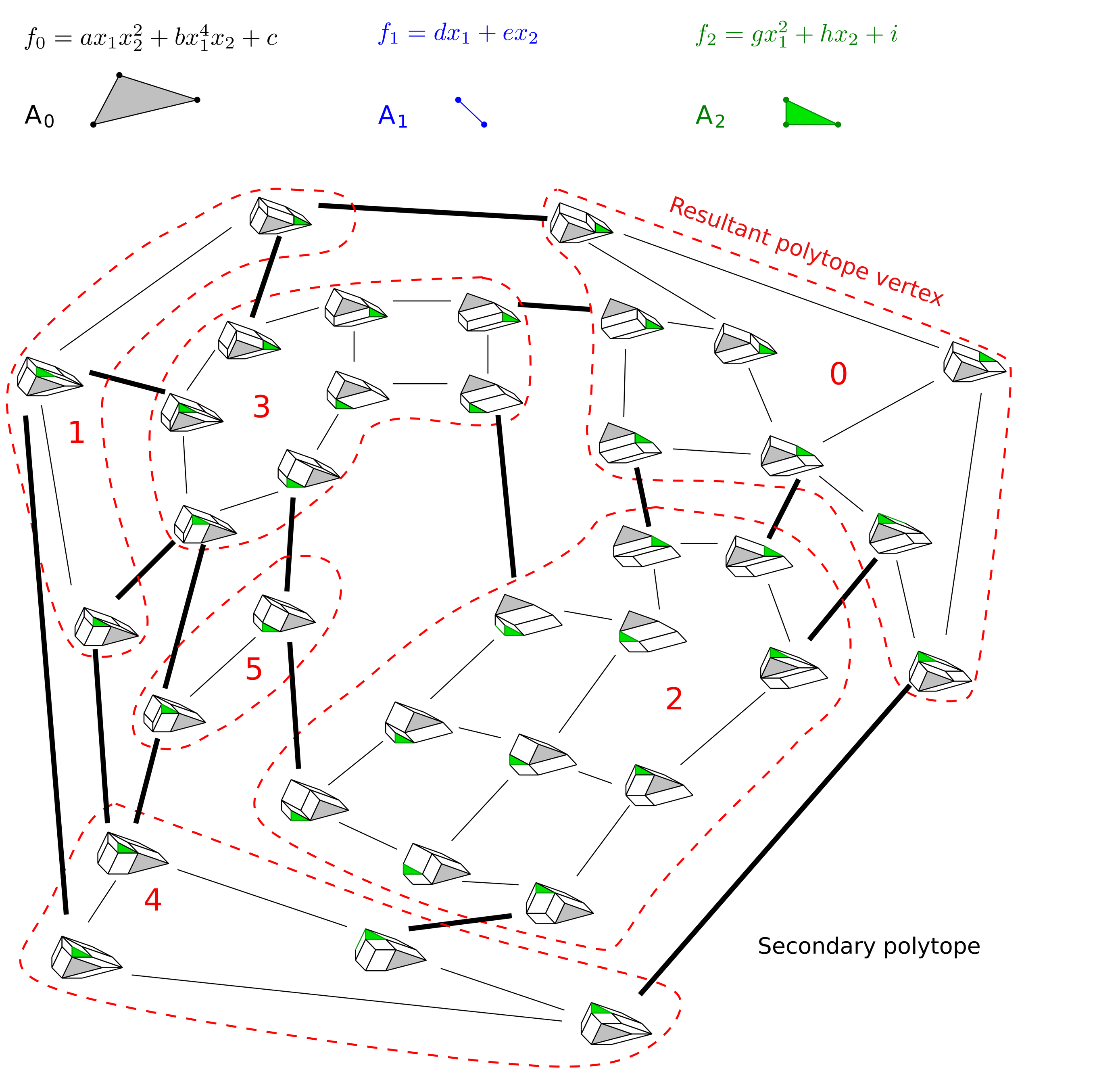
- The **secondary polytope** of a pointset $B \subseteq \mathbb{Z}^n$ is a polytope of dimension $|B| - n - 1$ in $\mathbb{R}^{|B|}$, namely $\Sigma(B)$, whose face poset is isomorphic to the poset of all regular subdivisions of B.
- Given $A_0, \dots, A_n \subseteq \mathbb{Z}^n$ the **(sparse) resultant** of a system of polynomials with supports A_i

$$f_0(x) = f_1(x) = \cdots = f_n(x) = 0, \qquad (2)$$

is the unique (up to sign) irreducible integer polynomial R in the $c_{i,a}$, which vanishes iff (2) has a solution in $(K^*)^n$.

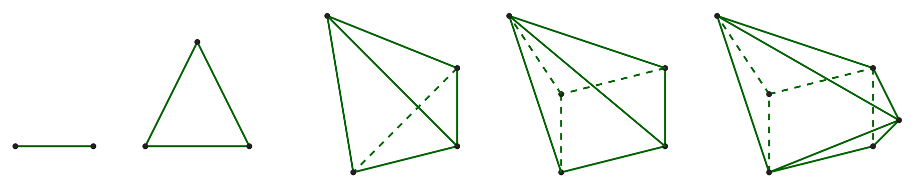
- There is a many-to-one correspondence from regular triangulations of $\mathcal{A}$ to vertices of $N(R)$.
- $N(R)$ is a Minkowski summand of $\Sigma(\mathcal{A})$.

An example



Existing work

- [GKZ90] Univariate case / general dimensional $N(R)$
- [St94] Multivariate case / up to 3 dimensional $N(R)$



Main result

Given $A_0, A_1, \dots, A_n \subset \mathbb{Z}^n$ with $N(R)$ of dimension 4. Then $N(R)$ are degenerations of the polytopes in following cases.

(i) All $|A_i| = 2$, except for one with cardinality 5, is a 4-simplex with f-vector $(5, 10, 10, 5)$.

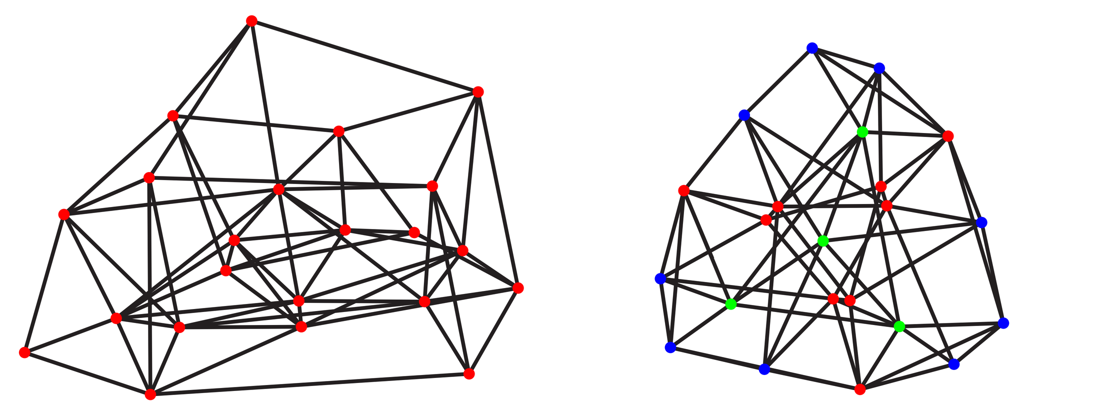
(ii) All $|A_i| = 2$, except for two with cardinalities 3 and 4, has f-vector $(10, 26, 25, 9)$.

(iii) All $|A_i| = 2$, except for three with cardinality 3, maximal number of ridges is 66 and of facets 22. Moreover, the maximal number of vertices is between 22 and 28, and of edges between 66 and 72. The lower bounds are tight.

▷ Degenarations can only decrease the number of faces.
▷ Focus on new case (iii), which reduces to $n = 2$ and each $|A_i| = 3$.
▷ Previous upper bound for vertices yields 6608 [St94].

Example of 4 dimensional resultant polytope

Vertex-graph (left) facet-graph (right; courtesy of M.Joswig) 4-dimensional $N(R)$ with f-vector $(22, 66, 66, 22)$ that maximizes the number of facets. Drawn with `polymake`.



Lower bounds

- f-vectors of 4-dimensional N(R)

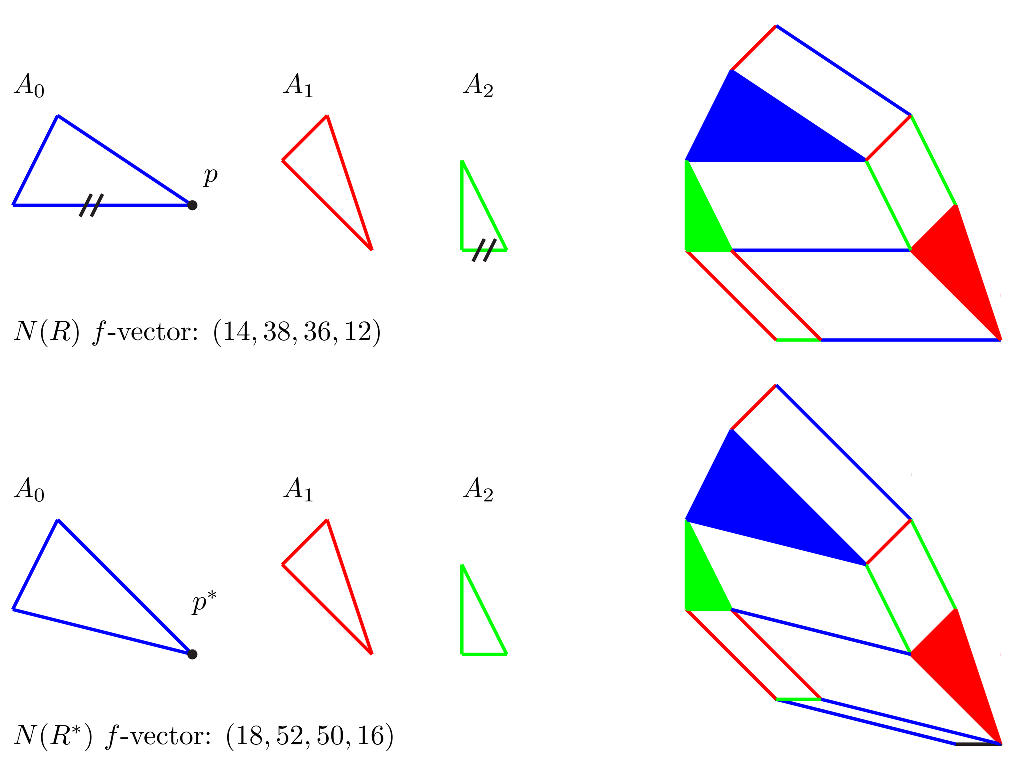
(5, 10, 10, 5)	(18, 53, 53, 18)
(6, 15, 18, 9)	(18, 54, 54, 18)
(8, 20, 21, 9)	(19, 54, 52, 17)
(9, 22, 21, 8)	(19, 55, 51, 15)
.	(19, 55, 52, 16)
.	(19, 55, 54, 18)
.	(19, 56, 54, 17)
(17, 49, 48, 16)	(19, 56, 56, 19)
(17, 49, 49, 17)	(19, 57, 57, 19)
(17, 50, 50, 17)	(20, 58, 54, 16)
(18, 51, 48, 15)	(20, 59, 57, 18)
(18, 51, 49, 16)	(20, 60, 60, 20)
(18, 52, 50, 16)	(21, 62, 60, 19)
(18, 52, 51, 17)	(21, 63, 63, 21)
(18, 53, 51, 16)	(22, 66, 66, 22)

- `respol` software [EFKP12]

▷ C++, CGAL (Computational Geometry Algorithms Library)
▷ `http://sourceforge.net/projects/respol`
▷ Alternative: `GFan` library, tropical geometry [JensenYu11]

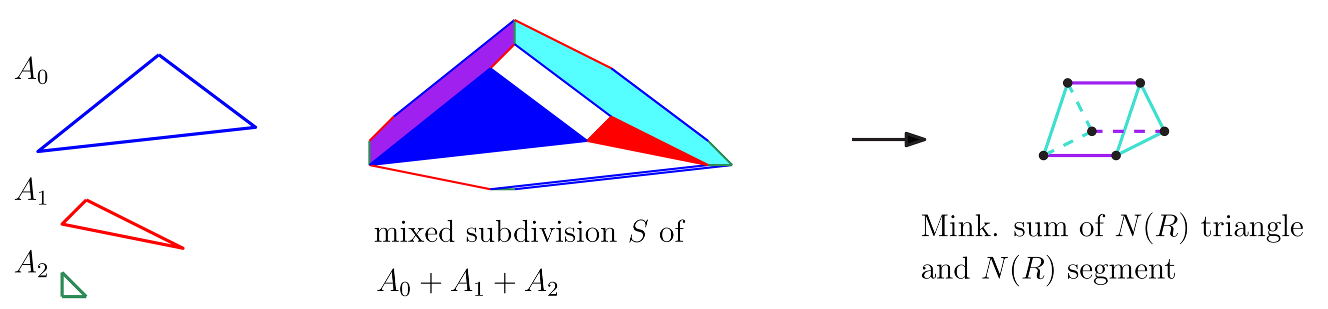
Upper bounds: Input genericity

For upper bounds on the number of $N(R)$ faces consider generic inputs, i.e. no parallel edges.



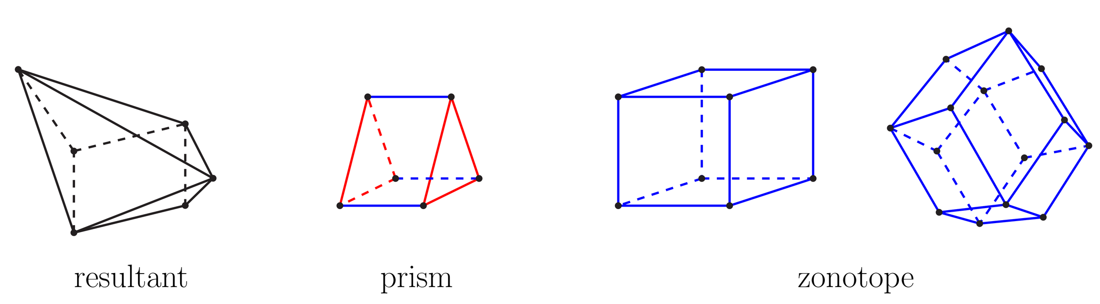
Upper bounds: Faces and subdivisions

A regular mixed subdivision S of $A_0 + A_1 + \cdots + A_n$ corresponds to a face of $N(R)$ which is the Minkowski sum of the resultant polytopes of the cells (subsystems) of S. [St94]



Facets of the resultant polytope

- resultant facet: 3-d N(R)
- prism facet: 2-d N(R) (triangle) + 1-d N(R)
- zonotope facet: sum of 1-d N(R)



Future work

- The maximum f-vector of a 4d-resultant polytope is $(22, 66, 66, 22)$.
- Explain symmetry of f-vectors of 4d-resultant polytopes.
- General dimension conjecture

$$f_0(d) \leq 3 \cdot \sum_{\|S\| = d-1} \prod_{i \in S} \tilde{f}_0(i)$$

where S is any multiset with elements in $\{1, \dots, d-1\}$, $\|S\| := \sum_{i \in S} i$, and $\tilde{f}_0(i)$ is the maximum number of vertices of a i-dimensional $N(R)$.

The only bound in terms of d is $(3d-3)^{2d^2}$ [St94], yielding $\tilde{f}_0(5) \leq 12^{50}$ whereas our conjecture yields $\tilde{f}_0(5) \leq 231$.

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