Polynomials and polytopes

- We consider \( n + 1 \) (Laurent) polynomials with fixed support sets \( A_i \subset \mathbb{Z}^n \), \( i = 0, \ldots, n \), in \( n \) unknowns \( x = (x_1, x_2, \ldots, x_n) \) over an algebraically closed field \( k \), with symbolic coefficients \( c_{i,a} \in A_i \).

- Newton polytope \( \Delta(f) \) of a polynomial \( f \) is the convex hull of its support set.

The Cayley trick

- Given points \( A_0, \ldots, A_n \subset \mathbb{Z}^n \), we define the point set

\[
A := \{ \sum_{i=0}^n f(A_i \times \{a_i\}) \} \subset \mathbb{Z}^{2n+1},
\]

where \( a_0, \ldots, a_n \) are an affine basis of \( \mathbb{R}^n \), \( a_0 = (0, \ldots, 0) \), \( a_i = (0, \ldots, 0, 1, 0, \ldots, 0) \), \( i = 1, \ldots, n \).

- A subdivision \( S \) of \( A_0 + A_1 + \cdots + A_n \) is mixed when its cells have expressions as Minkowski sums of point subsets in \( A_i \). Right minimal mixed subdivisions of \( A_0 + \cdots + A_n \) are in bijection with regular triangulations of \( A \).

Secondary and resultant polytopes

- The secondary polytope of a point set \( B \subset \mathbb{Z}^n \) is a polytope of dimension \( |B| - n - 1 \) in \( \mathbb{R}^{|B|} \), namely \( \Sigma(B) \), whose face poset is isomorphic to the poset of all regular subdivisions of \( B \).

- Given \( A_0, \ldots, A_n \subset \mathbb{Z}^n \), the (sparse) resultant of a system of polynomials with supports \( A_i \)

\[
\delta_i(x) = \delta_i(x_1) \cdots = \delta_i(x_n) = 0.
\]

- is the unique (up to sign) irreducible integer polynomial \( R \) in the \( c_{i,a} \) which vanishes iff \( f(2) \) has a solution in \( \mathbb{K}^n \).

- There is a many-to-one correspondence from regular triangulations of \( A \) to vertices of \( \Sigma(R) \).

\( \Sigma(R) \) is a Minkowski sum of \( \Sigma(A) \).

Existing work

- \[GKZ90\] Univariate case / general dimensional \( N(R) \)

- \[St94\] Multivariate case / up to 3 dimensional \( N(R) \)

Lower bounds

- \( f \)-vectors of 4-dimensional \( N(R) \)

- (9, 10, 5)

- (6, 15, 8)

- (6, 20, 11)

- (9, 22, 8)

- (17, 49, 16)

- (17, 49, 17)

- (17, 50, 17)

- (18, 51, 18)

- (18, 52, 17)

- (18, 55, 16)

- (22, 66, 22)

- Exapol software \[EFKP12\]

Upper bounds: Input genericity

For upper bounds on the number of \( N(R) \) faces consider generic inputs, i.e. no parallel edges.

\( N(R) \)-vector: \((18, 53, 51, 16)\)

\( N(R^*) \)-vector: \((14, 38, 36, 12)\)

\( p \) \( p^* \) \( A_0 A_1 A_2 \)

Future work

- The maximum \( f \)-vector of a 4d-resultant polytope is \((22, 66, 66, 22)\).

- Explain symmetry of \( f \)-vectors of 4d-resultant polytopes.

- General dimension conjecture

\[
\delta(d) \leq 3 \sum_{|S|=d-1} \prod_{c \in S} \delta(c)
\]

where \( S \) is any multiset of elements in \( \{1, \ldots, d-1\} \), \( |S| = \sum_{i=1}^{d-1} |c_i| \), and \( \delta(c) \) is the maximum number of vertices of a \( 1 \)-dimensional \( N(c) \).

The only bound in terms of \( d \) is \((3d - 3)!! \) \[St94\], yielding \( \delta(3) \leq 231 \) whereas our conjecture yields \( \delta(3) \leq 231 \).

References

- \[DEF13\] A. Dickenstein, I.Z. Emiris, V. Fisikopoulos.


- An output-sensitive algorithm for computing projections of resultant polytopes.

- UCGA special issue on ACM SCGC 2012.

- \[GKZ90\] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky.


- \[JensenYu11\] A. Jensen and J. Yu.


- \[St94\] B. Sturmfels.