Regular triangulations and resultant polytopes

Ioannis Z. Emiris* Vissarion Fisikopoulos*† Christos Konaxis*

Abstract

We describe properties of the Resultant polytope of a given set of polynomial equations towards an output-sensitive algorithm for enumerating its vertices. In principle, one has to consider all regular fine mixed subdivisions of the Minkowski sum of the Newton polytopes of the given equations. By the Cayley trick, this is equivalent to computing all regular triangulations of another point set in higher dimension. However, the number of all regular triangulations is generally much larger than that of the vertices of the Resultant polytope, as illustrated by our experiments [3]. Thus, we study output-sensitive methods by defining classes of subdivisions, called configurations, which yield the same resultant vertex. Moreover, we offer algorithmic versions of certain results by Sturmfels [11], regarding the edges of the Resultant polytope. Lastly, we settle some easy cases, and discuss harder examples.

1 Introduction

We are interested in algorithms that compute the Newton polytope of the Resultant, called Resultant polytope, of a given set of polynomial equations. Resultants are fundamental objects in polynomial equation solving [12], and in implicitizing parametric (hyper)surfaces [2]. In fact, a projection of the resultant polytope yields the Newton polytope of the (unknown) implicit equation, thus reducing implicitization to a problem in linear algebra. One approach is to compute the regular fine mixed subdivisions of the Minkowski sum of the Newton polytopes of the given equations. Another is based on tropical geometry, e.g. [12, ch.9].

These regular fine mixed subdivisions correspond by Cayley trick to the regular triangulations of a point set A. For each point set A, its Secondary polytope's vertices correspond to the regular triangulations of A and there are output-sensitive that enumerate them [6, 10]. However, the number of vertices of a Secondary polytope can be exponential in $|A|$ and there is a many to one correspondence of Secondary vertices to the vertices of the Resultant polytope, illustrated by our experiments [3]. On the other hand, for the Resultant polytope, we only know a weak exponential upper bound on the number of vertices [11, prop.6.1]. The above results force us to focus on output-sensitive algorithms that enumerate classes of subdivisions which yield the same resultant vertex, without enumerating the entire Secondary polytope. We present our work in progress to this end. We offer algorithmic versions of certain results in [11] regarding the edges of the Resultant polytope. Lastly, we settle some easy cases, and discuss harder examples.

2 Triangulations, Mixed Subdivisions, and Polynomial Systems

Let A be a point set in $\mathbb{R}^d$. A *polyhedral subdivision* of A is a collection of subsets of A, the cells of the subdivision, such that the union of the cells' convex hulls equals the convex hull of A and every pair of convex hulls of cells intersect at a common face. A polyhedral subdivision is *regular* if it can be obtained as the projection of the lower hull of the lifted point set A, for some lifting to $\mathbb{R}^{d+1}$. A *triangulation* T is a polyhedral subdivision of A, whose cells are all simplices. *Circuits* are the minimum affinely dependent subsets of a point set that have exactly two triangulations. A *bistellar flip* transforms one triangulation to another. Let T be a triangulation of A and $Z_+ \subseteq T$ the triangulation of a circuit $Z \subseteq A$. We say T is *supported* on Z if, by changing the current triangulation Z_+ of Z to the other, denoted Z_-, we obtain another triangulation T'. This is a bistellar flip of T supported on Z. If $|A| = n$, its *Secondary polytope* $\Sigma(A)$ has dimension $n-d-1$, its vertices correspond to the regular triangulations of A, and its edges to bistellar flips [5, 8].

Let $A_0, \ldots, A_k$ be point sets in $\mathbb{R}^d$ and $\overline{A} = A_0 + \cdots + A_k$ their Minkowski sum. A subset of $\overline{A}$ is called *Minkowski cell* if it can be written as $F_0 + \cdots + F_k$ for $F_i \subseteq A_i$. A Minkowski cell is *fine* if all F_i are affinely independent and $\sum_{i=1}^{k} dim(CH(F_i)) = d$. When $k = d$, a Minkowski cell is *i-mixed* if it is a Minkowski sum of k edges and a vertex, i.e., $|F_j| = 2$ for $j \neq i$, $|F_i| = 1$. When $k = d-1$, a Minkowski cell is *mixed* if it is a Minkowski sum of edges. A regular polyhedral subdivision of $\overline{A}$ is a *regular fine mixed subdivision* if all its cells are Minkowski and fine. From now on we consider all mixed subdivisions to be regular and fine, and focus on $k = d$, unless otherwise noted. This is the most important case because it covers system

*Department of Informatics & Telecommunications, National and Kapodistrian University of Athens, Greece.
†Contact author (vfisikop@di.uoa.gr)

solving and implicitization; implicitization of surfaces in $\mathbb{R}^3$ corresponds to $k = d = 2$.

Let $f = f_0, \dots, f_k$ be a polynomial system on k variables. The support $A_i \in \mathbb{N}^k$ of f_i is the set of its exponent vectors corresponding to nonzero coefficients. For any subset $J \subset \{0, \dots, k\}$, let $r(J)$ denote the rank of the affine lattice generated by $\sum_{j \in J} A_j$. We assume that, for $I = \{0, \dots, k\}$, $r(I) = |I| - 1$, and $r(J) \geq |J|$ for any proper subset $J \subset I$. The *Newton polytope* $N(f_i)$ of a polynomial f_i is the convex hull of its support. The (sparse) Resultant R of f is a polynomial on the coefficients of f such that $R = 0$ iff f has a solution in $(\mathbb{C}^*)^k$. It generalizes the determinant of an overconstrained linear system and the Sylvester resultant of two univariate polynomials. We call $N(R)$ the *Resultant polytope* and *extreme term* of R a monomial which corresponds to a vertex of $N(R)$.

Proposition 1 [11, thm.2.1] *Following the above notation and assumptions, given a system f and a mixed subdivision of the Minkowski sum of its supports, we get an extreme term of the resultant R equal to*

$$\pm \prod_{i=0}^{k} \prod_{\sigma} c_{iF_i}^{vol(\sigma)},$$

where $\sigma = F_0 + \cdots + F_k$ is an i-mixed cell and $vol(\cdot)$ denotes Euclidean volume.

By the Cayley trick [5], there is a point set $C(A_0, \dots, A_k) \subset \mathbb{R}^{d+k}$ s.t. all mixed subdivisions of $A = A_0 + A_1 + \cdots + A_k$ are in 1-1 correspondence with the regular triangulations of $C(A_0, A_1 \dots, A_k)$. Hence, one can obtain the $N(R)$ vertices by enumerating all vertices of the corresponding Secondary polytope $\Sigma(A_0, A_1 \dots, A_k)$. Moreover, $N(R)$ is a Minkowski summand of $\Sigma(A_0, A_1 \dots, A_k)$ [11]. Methods to enumerate regular triangulations have been proposed in [6, 10], and are experimented with in [3]. But we can do better.

When $k = d - 1$, mixed cell *configurations* are the equivalence classes of mixed subdivisions with the same mixed cells. These are defined, along with a definition of flips between these classes, in [9].

When $k = d$, we focus on the i-mixed cells in order to compute the vertices of $N(R)$. In [7], there is an extension of mixed cells configurations to classes containing the same i-mixed cells for all $i \in \{0, \dots, k\}$, called i-mixed cell configurations. It turns out that this notion is similar to the I-mixed cell configurations of [1]. We now characterize the flips between i-mixed cell configurations, and generalize the flip defined in [9] between mixed cell configurations.

We shall say that a circuit Z of a triangulation T supported on Z, *involves* an i-mixed cell $F_0 + \cdots + F_k$, if the cell $C(F_0, \dots, F_k)$ of T does not belong to the triangulation obtained by flipping on Z.

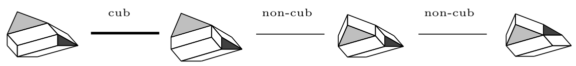

Fig. 1: An example of a cubical and two non cubical flips.

Theorem 2 *([7]) Let $Z = (Z_0, \dots, Z_k)$ be a circuit and T a triangulation supported on Z. Suppose that Z involves an i-mixed cell $F_0 + \cdots + F_k$. Then, there exists $r \in \{0, \dots, k\}$, and $c \in A_r$ s.t. for all $i \neq r$, $Z_i = F_i$ or $Z_i = \emptyset$, and $Z_r = F_r \cup \{c\}$ or $Z_r = \{v_r, c\}$, where v_r is a vertex of edge F_r.*

A flip on a circuit as described in this theorem destroys at least one i-mixed cell leading to a new i-mixed cell configuration. Moreover, we can check efficiently if a circuit satisfies the conditions of th. 2 by examining only the cardinalities of the sets Z_i. An algorithm using these flips enumerates only the i-mixed cell configurations, without enumerating all mixed subdivisions, which are more numerous.

The Ξ *polytope* is defined in [1] for $k \leq d-1$. In particular, when $k = d - 1$, Ξ has vertices corresponding to mixed cell configurations, and edges corresponding to flips between them. Based on this, we define Ξ in the case $k = d$ to have vertices corresponding to i-mixed cell configurations. Clearly, Ξ lies, in terms of number of vertices, between the Secondary polytope and $N(R)$. In the sequel, Ξ or $\Xi(A_0, A_1, \dots, A_k)$ refers to this polytope.

3 R-equivalent Classes

By prop. 1, several mixed subdivisions may produce the same extreme term of the Resultant. We call these subdivisions *R-equivalent*. Similarly, two subdivisions may lead to the same extreme term, even if they belong to the same i-mixed cell configuration. These R-equivalent classes correspond to the vertices of $N(R)$. There are some flips that connect two subdivisions in different R-equivalent classes, hence they correspond to the edges of $N(R)$.

Sturmfels [11, thm.5.2] calls these flips *cubical*. Consider the union of cells affected by one such flip. If the union, lifted generically to $\mathbb{R}^{d+1}$, forms an affine cube, i.e. equals the Minkowski sum of $k + 1$ edges, then the flip is cubical and consists in replacing the "bottom" subdivision by the "top" subdivision, or vice versa (fig. 1, fig. 2). However, this definition of cubical flips is not algorithmically efficient, so we provide a more algorithmic characterization.

Let us start with the generic case, where every two faces of the same dimension in two different $CH(A_i)$ are not parallel.

Lemma 3 *Let S be a mixed subdivision of $A_0 + \cdots + A_k$. Then S has a cubical flip iff there exists a set $\{C_0, \dots, C_k\}$ of i-mixed cells $C_i = F_0 + \dots + a_i + \cdots +$*

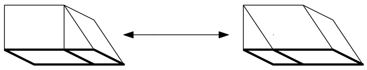

Fig. 2: A degenerate cubical flip; two (horizontal bold) edges from different A_i's are parallel.

F_k, for $i = \{0, 1, \ldots, k\}$, where $a_i \in F_i \subseteq Ai, |F_i| = 2$, such that, if $C = \bigcup_{i=0}^{k} C_i$, then $C = F_0 + \cdots + F_k$. We say S is supported on C. The cubical flip on S consists of substituting, in every C_i, point a_i with $F_i - \{a_i\}$.

If the generic position assumption does not hold, lem. 3 does not hold, so we generalize this characterization using triangulations. Recall that the set C of i-mixed cells corresponds by Cayley trick to a set Z of simplices and a flip between two mixed subdivisions is a flip between the two corresponding triangulations. Genericallly, C has $k+1$ cells and Z has $k+1$ simplices. The union of these simplices contains $2k+2$ points in a space of dimension $k+d$. If $d = k$, this union of simplices is a circuit. In degenerate cases, there may exist lower dimensional circuits and C may have $< k+1$ cells. As an illustration compare the generic example (fig. 1), where C has 3 cells, with the degenerate example (fig. 2), where C has only 2 cells.

Theorem 4 *Let S be a mixed subdivision of $A_0 + \cdots + A_k$, and T the corresponding triangulation w.r.t. Cayley's trick. Then S has a cubical flip if there exists a set $C = \bigcup_{i=0}^{k} C_i \subseteq S$ of i-mixed cells, as in lem. 3 and, additionally, the corresponding set Z of simplices in T supports a bistellar flip. The cubical flip on S is the bistellar flip of T supported on Z.*

The mapping of cubical flips edges of $N(R)$ is many to one. When a cubical flip is supported on set C, we say that the edge is of type C. Many cubical flips may be supported on the same set C. The types of all Resultant edges can be easily enumerated: they are all possible resultant polytopes of subsets of A_i's with cardinality two. This enumeration also yields the corresponding edge direction, i.e. the difference vector between the two endpoints of $N(R)$. More generally, all faces of $N(R)$ are Minkowski sums of Resultant polytopes corresponding to subsystems of $A_0, \ldots, A_k$. Conversely, every resultant polytope defined on subsets of the A_i's appears as Minkowski summand on some face of $N(R)$ [11].

Example 5 Let A_0= $\{(0,0), (1,2), (4,1)\}$, A_1= $\{(0,1), (1,0)\}$, A_2= $\{(0,0), (0,1), (2,0)\}$, which satisfy the general position assumption. The Secondary polytope of $C(A_1, A_2, A_3)$ is depicted in fig. 3 (left). One can see the R-equivalent classes (dotted) as well as the cubical flips (bold) which connect these classes. All the other flips (non bold) are non-cubical flips. The Resultant polytope can be seen as the polytope with R-equivalent classes as vertices and cubical flips as edges. To each Resultant vertex corresponds one or more mixed subdivisions, and to each edge one or more cubical flips. Here, $\Sigma(C(A_0, A_1, A_2) = \Xi(A_0, A_1, A_2)$ with 36 vertices; $N(R)$ has 6 vertices, and 11 edges corresponding to 9 different cubical flips (fig. 3 right) which are all generic.

4 Secondary, Ξ, and Resultant Polytopes

Let Σ be the Secondary polytope of $C(A_0, A_1, \ldots, A_k)$, Ξ the polytope $\Xi(A_0, A_1, \ldots, A_k)$, and $N(R)$ the Resultant polytope. We offer a case study on these polytopes, and focus on $d = k$.

When $d = k = 1$, every Minkowski cell is an edge, i.e., a sum of an edge and a vertex, thus an i-mixed cell. Then $\Sigma = \Xi$, and they are generally larger than $N(R)$. The number of vertices of $N(R)$ is $\binom{m_0+m_1-2}{m_0-1}$ [4].

For arbitrary d, k, if all $|A_i| \leq 3$, then $\Sigma = \Xi$. To see this, recall that for any fine Minkowski cell $F = \sum_{i=1}^{k} F_i$, it holds that $\sum_{i=1}^{k} dim(F_i) = d$. So, F is not i-mixed iff for some F_i, we have $dim(F_i) > 1$. Since $|A_i| \leq 3$, by the pigeonhole principle, every non i-mixed cell is a sum of $k-2$ edges, two vertices and a triangle. So the union of every pair of non i-mixed cells can be written uniquely.

The smallest case that this does not hold is when there exists i s.t. $|A_i| = 4$, and $|A_j| \leq 3$, $\forall j \neq i$.

Example 6 An instance of the smallest case, for $d = k = 2$, is A_0= $\{(0,0), (0,1), (2,0), (2,1)\}$, A_1= $\{(0,0), (1,1), (2,0)\}$, A_2= $\{(0,0), (0,1)\}$, where Σ, Ξ, and $N(R)$ have, resp., $122, 98$, and 8 vertices.

For arbitrary d, k, if for all i, $|A_i| = 2$, then $\Sigma = \Xi = N(R)$. This is the case where all flips are cubical, every Minkowski cell is a sum of edges, called zonotope, and the mixed subdivisions are zonotopal tilings. The above discussion proves the following.

Lemma 7 *If $d = k = 1$, or for all i, $|A_i| \leq 3$, then $\Sigma = \Xi$ and they are at least as large as $N(R)$. If for all i, $|A_i| = 2$, then $\Sigma = \Xi = N(R)$.*

In addition to the case analysis above, we offer relevant experimental results in [3]. In particular, we consider some (highly) nontrivial examples corresponding to the implicitization of a parametric sphere [2].

Example 8 The A_i's are $\{(0,0), (0,2), (2,0), (2,2)\}$, $\{(0,0), (1,0), (0,2), (2,0), (1,2), (2,2)\}$, $\{(0,0), (0,1), (0,2)\}$, and Σ, Ξ and $N(R)$, resp. have 76280, 32076 and 95 vertices.

The A_i's are $\{(0,0), (1,0), (0,2), (2,0), (1,2), (1,2)\}$, $\{(0,0), (0,2), (1,1), (2,0), (2,2)\}$, $\{(0,0),$

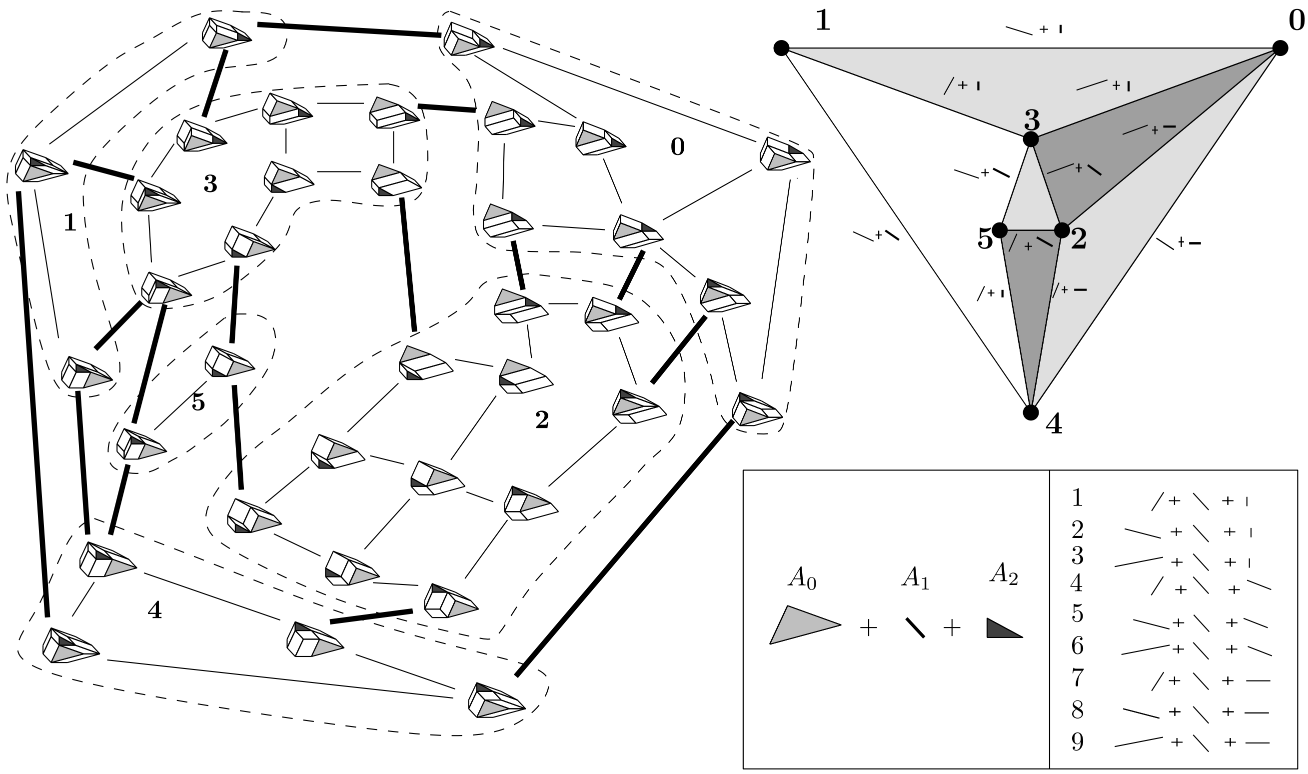

Fig. 3: [left] The Secondary polytope of example 5 with the R-equivalent classes (dotted) and the cubical flips (bold). [right-up] The Resultant polytope of example 5. Each edge is a cubical flip; note that the same cubical flip may appear in two different edges (9 flips in 11 edges). [right-botom] The input point sets and the cubical flips.

$(2,0)\}$ and Σ, Ξ and $N(R)$ have resp. 104148, 43018 and 21 vertices.

Our ultimate goal is an algorithm to enumerate all vertices of $N(R)$ without enumerating the entire Σ or Ξ. To this end we need a unique representation of the resultant vertices and some kind of flip based on the cubical flip. At present, cubical flips do not suffice because there are cases where it is not clear how to obtain one vertex from another just with cubical flips (see fig.3).

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