Experimental Study of the Ehrhart Interpolation Polytope

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Abstract. In this paper we define a family of polytopes called Ehrhart Interpolation Polytopes with respect to a given polytope and a parameter corresponding to the dilation of the polytope. We experimentally study the behavior of the number of lattice points in each member of the family, looking for a member with a single lattice point. That single lattice point is the h* vector of the given polytope. Our study is motivated by efficient algorithms for lattice point enumeration.

1 Introduction

A fundamental problem in discrete and computational geometry is to efficiently count or enumerate the lattice points of a polytope. Let $P \subseteq \mathbb{R}^d$ be a full dimensional polytope and L a lattice in $\mathbb{R}^n$. For any positive integer t, let $tP = \{tp : p \in P\}$ be the t-fold dilation of P and $\mathcal{L}_P(t) = \#(tP \cap L)$ be the counting function for the number of lattice points contained in tP. Now, let P be a lattice polytope, i.e., the vertices of P are lattice points. It is known, due to Ehrhart [2], that there exist rational numbers $a_0, \ldots, a_d$ such that $\mathcal{L}_P(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t + a_0$. $\mathcal{L}_P(t)$ is called the Ehrhart polynomial of P. Note that the degree of $\mathcal{L}_P(t)$ is equal to the dimension of polytope P. Instead of considering $\mathcal{L}_P(t)$ in the monomial basis, we can also view it as a polynomial in the binomial basis $\left\{\binom{t+d-i}{d}\right\}_{i=0,\ldots,d}$ of polynomials of degree up to d. Then from [6], we have that

$$\mathcal{L}_P(t) = \sum_{j=0}^{d} h_j^* \binom{t+d-j}{d} \text{ and } h_j^* \in \mathbb{N} \tag{1}$$

where $(h_0^*, h_1^*, \ldots, h_d^*)$ is the h^* vector of P.

The h^* vector was the subject of many studies in the last decades [1,6,8]. A lot of interesting results exist, but we will only mention the ones relevant for the definition of the Ehrhart Interpolation Polytope (see Sect. 2).

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2 The Ehrhart Interpolation Polytope

In this section, we define the main object of our study, the Ehrhart Interpolation Polytope. For this we will use the H^* polyhedron. The main idea is that there is a number of known linear inequalities for h^* vectors, thus defining a polyhedron. This polyhedron is contained in the non-negative orthant of $\mathbb{R}^{d+1}$, since h^* vectors are non-negative.

Definition 1 *(H^* **polyhedron**). Given $d \in \mathbb{N}^*$, let*

$$H^* = \left\{ \begin{array}{lll} x \in \mathbb{R}^{d+1} : & & \\ x_0 = 1 & & [2] \\ x_i \geq 0 & ,0 \leq i \leq d & [6] \\ x_i \geq x_1 & ,2 \leq i < d & [1] \\ x_d + x_{d-1} + \cdots + x_{d-i} \leq x_0 + x_1 + \cdots + x_{i+1} & ,0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor & [1] \\ x_0 + x_1 + \cdots + x_i \leq x_d + x_{d-1} + \cdots + x_{d-i} & ,0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor & [1] \\ x_1 + x_2 + \cdots + x_i \leq x_{d-1} + x_{d-2} + \cdots + x_{d-i} & ,0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor & [8] \\ x_{d-1} + x_{d-2} + \cdots + x_{d-i} \leq x_2 + x_3 + \cdots + x_{i+1} & ,0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor & [8] \end{array} \right\}$$

Lemma 1. *All h^* vectors of d-dimensional polytopes with at least one interior lattice point are lattice points in H^*.*

For special cases of d it is possible to have more constraints that could yield a more refined H^*.

The polyhedron H^* depends only on the dimension d. If we are given a polytope $P \subseteq \mathbb{R}^d$, we can obtain upper bounds for the h^* vector of P. In [7], Stanley proves the following monotonicity theorem.

Theorem 1 **([7]).** *If P and Q are polytopes in $\mathbb{R}^n$ and $P \subseteq Q$, then $h^*_{P,i} \leq h^*_{Q,i}$, where h^*_P and h^*_Q are the h^* vectors of P and Q respectively.*

The above result could yield upper bounds for the h^* vector of a given polytope P by constructing the smallest hypercube C containing P. The h^* vectors of lattice hypercubes are easy to compute and this way we bound from above all coordinates of the h^* vector of P as $h^*_{P,i} \leq h^*_{C,i}$ for $0 \leq i \leq d$. We use the constraints coming from the bounding hypercube together with the one defined by Eq. 1 to define the Ehrhart Interpolation Polytope.

Definition 2 (Ehrhart Interpolation Polytope). *Given a polytope $P \subseteq \mathbb{R}^d$ with at least one lattice point in its interior and $t \in \mathbb{N}^*$, we define the Ehrhart Interpolation Polytope of P in dilation t*

$$\mathcal{E}_P(t) = \left\{ \begin{array}{l} x \in \mathbb{R}^{d+1} : \\ x \in H^*, \\ \sum_{i=0}^{d} x_i \binom{t+d-i}{d} = \mathcal{L}_P(t), \\ x_i \leq h^*_{C,i},\ 0 \leq i \leq d \end{array} \right\} \tag{2}$$

where $C \subseteq \mathbb{R}^d$ is the smallest cube containing P.

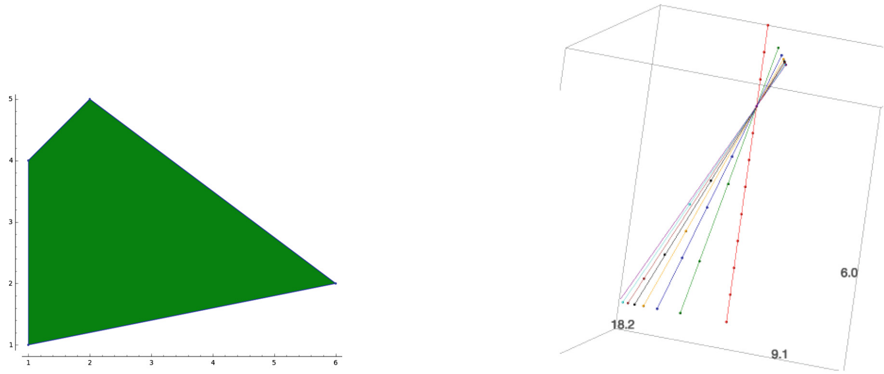

Fig. 1. The polytope P (left) and the associated $\mathcal{E}_P(t)$ (right) from Example 1.

Note that $\mathcal{E}_P(t)$ is indeed a bounded polyhedron, since it is contained in the intersection of the positive orthant with a hyperplane whose normal vector is a strictly positive vector (containing binomial coefficients). Moreover, observe that the h^* vector of P is contained in $\mathcal{E}_P(t)$ for all $t \in \mathbb{N}^*$ by construction, for all polytopes P containing at least one lattice point in their interior. Generically, H^* and $\mathcal{E}_P(t)$ will have dimension d and $d-1$ respectively.

Example 1. Let $P \subseteq \mathbb{R}^2$ be the convex hull of the points $(1,1),(1,4),(2,5),(6,2)$. Then the Ehrhart Interpolation Polytope $\mathcal{E}_P(t)$ is an 1-dimensional polytope in $\mathbb{R}^3$.

Figure 1 depicts P and $\mathcal{E}_P(t)$ for $t = 1,2,\ldots,8$. For $t = 1,2,\ldots,8$, there are $12,3,6,2,3,2,3,1$ lattice points in each segment respectively. For $t = 8$ (purple), $\mathcal{E}_P(t)$ contains a single lattice point. This lattice point is the h^* vector of the polytope P. Note that for dilations greater than 8, it is possible to have more than one lattice points, see Sect. 3.

The single lattice point in $\mathcal{E}_P(8)$ is $(1,12,9)$. The binomial basis for polynomials in dimension 2 is $\left\{\frac{(t+1)(t+2)}{2}, \frac{t(t+1)}{2}, \frac{(t-1)t}{2}\right\}$. By evaluating Eq. 1 we get the Ehrhart polynomial of P which is $11t^2 + 3t + 1$.

3 Experiments and Statistics

We experimentally study the number of lattice points of the $\mathcal{E}_P(t)$ as a function of the dilation t. The goal is to find a dilation t such that $\mathcal{E}_P(t)$ contains a single lattice point. Then, that lattice point is the h^* vector of P. Our experiments indicate that as we increase the dilation, after some point, the h^* vector becomes a vertex of the integer hull of the Ehrhart Interpolation Polytope. Moreover, experimental evidence indicates that after a certain threshold, for some dilations the integer hull of the Ehrhart Interpolation Polytope is 0-dimensional, i.e., a single point. This can already be observed in the low dimensions; see Example 1.

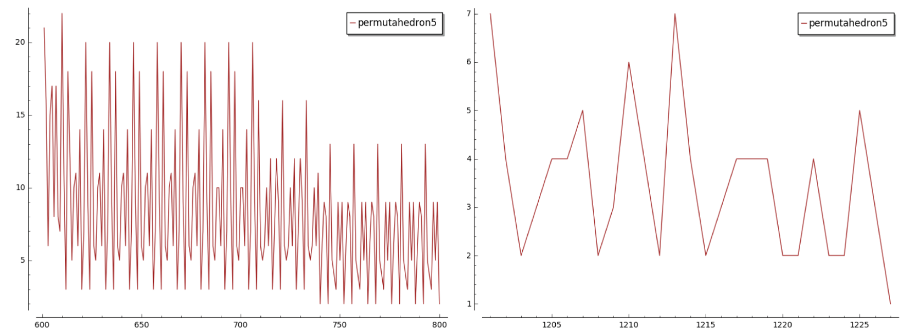

Fig. 2. The number of lattice points in the Ehrhart Interpolation Polytope for dilations 600–800 (left) and dilations 1200–1227 (right). Dilation 1227 contains one lattice point.

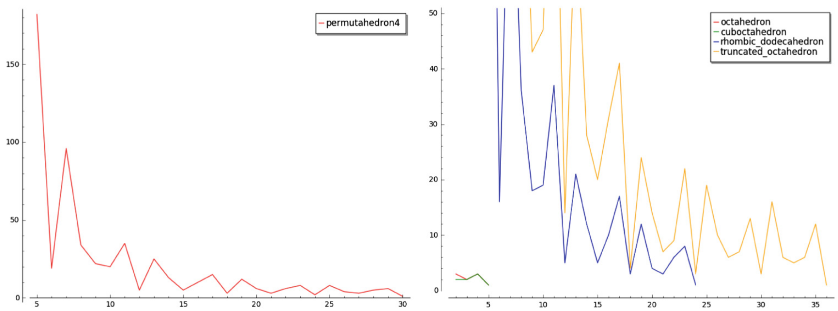

Fig. 3. Number of lattice points in the Ehrhart Interpolation Polytope for the permutahedron in dimension 4 (left) and for some solids (right) for dilations 5 to 30.

In this section we present some statistics about the number of lattice points in the Ehrhart Interpolation Polytope.

For our study we concentrate on cross polytopes, permutahedra, random simplices, and zonotopes in dimensions 3, 4, and 5, as well as Platonic solids that are lattice polytopes. For the exploration presented here we used the computer algebra system Sage [5].

We first focus on the permutahedron in dimension 5, to show some observations that hold for other families of polytopes as well. In Fig. 2, we see that the number of lattice points in successive dilations exhibits a periodic behavior. The lowest point during a period is related to the dimension of the polytope. Note that the permutahedron in dimension d is a $d-1$-dimensional polytope.

In Fig. 3, we can see that the Ehrhart Interpolation Polytope of the permutahedron in dimension 4 contains a single lattice point in dilation 30 on the left. On the right, we see the behavior of some Platonic Solids that are lattice polytopes.

Regarding timings, all above experiments have been performed on a personal computer in order of seconds.

4 Conclusion

The purpose of this short paper is to start a discussion on the study of Ehrhart Interpolation Polytopes. Our preliminary results indicate that it should be interesting to study more their combinatorial properties and provide rigorous experimental or analytical results.

Our original motivation for the definition of Ehrhart Interpolation Polytope was the computation of Ehrhart polynomials, using good approximations of the volume in large dilates of the polytope. Matthias Köppe [4] suggested to use Integer Linear Programming for finding the h^* vector in the Ehrhart Interpolation Polytope.

Interestingly, finding the Ehrhart polynomial reduces to a non-convex optimization problem, namely finding the minimum of the function counting the number of lattice points of a polytope varying dilation t. Finally, practical volume approximation algorithms [3] can be applied to yield bounds on the number of lattice points that could be used to refine $\mathcal{E}_P(t)$.

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