The Newton polytope of the Sparse Resultant

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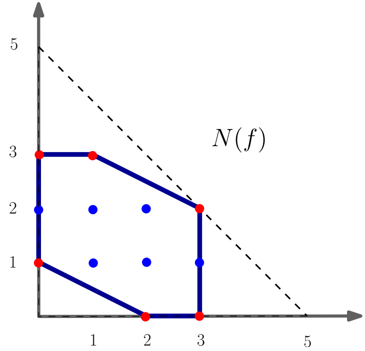
Newton polytope

Definition

Given polynomial $f \in K[x_1, \ldots, x_n]$ the Newton polytope $N(f)$ of f is the convex hull of its support set i.e. exponent vectors of monomials with non-zero coefficient.

$f(x_1, x_2) = 8x_2 + x_1x_2 - 24x_2^2 - 16x_1^2 + 220x_1^2x_2 - 34x_1x_2^2 - 84x_1^3x_2 + 6x_1^2x_2^2 - 8x_1x_2^3 + 8x_1^3x_2^2 + 8x_1^3 + 18x_2^3$



Sparse resultant

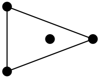
Definition

Given are polynomials $f_0, f_1, \ldots, f_n \in K[x_1, \ldots, x_n]$, s.t. the supports define an essential family $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$, i.e. the A_i generate $\mathbb{Z}^n$ and any k-subset generates a sublattice of dimension $\geq k$.
The system's (sparse) resultant R is the polynomial in the system's coefficients, defined up to sign, which vanishes iff the polynomials have a common root in $(\mathbb{C}^*)^n$.
The resultant polytope $N(R)$ is the Newton polytope of R.

$f_0(x) = ax^2 + b$ A_0

$f_1(x) = cx^2 + dx + e$ A_1

$R(a, b, c, d, e) = ad^2b + c^2b^2 - 2caeb + a^2e^2$ $N(R)$



Birkhoff polytope

Linear polynomials

A_0

$$f_0(x,y) = ax + by + c$$

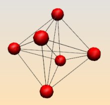
A_1

$$f_1(x,y) = dx + ey + f$$

A_2

$$f_2(x,y) = gx + hy + i$$

$N(R)$

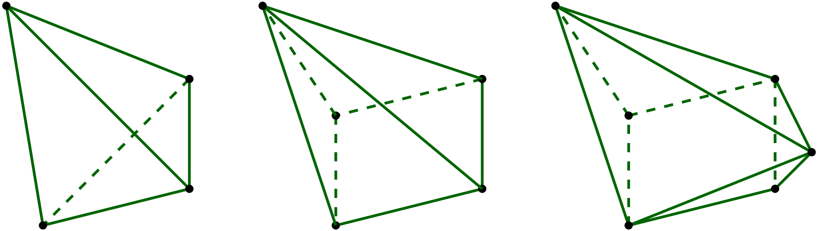


4-dimensional Birkhoff polytope

$$R(a,b,c,d,e,f,g,h,i) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Motivation

- Algebraic: complexity of resultant polynomial
- Geometric: generalize Birkhoff polytopes; faces are Minkowski sums of resultant polytopes
- Applications: support computation → interpolate implicit equation of parametric hypersurface, resultant, discriminant.

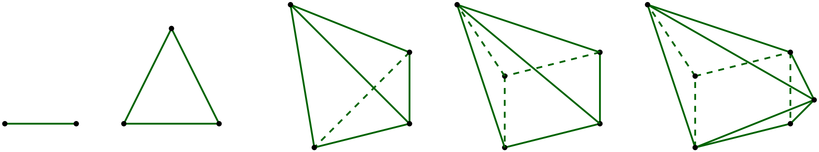


Examples of resultant polytopes

Previous work

- [GKZ'90] Univariate case, general-dimensional $N(R)$:
  The A_i are multisets from $\mathbb{Z}$: $|A_0| = k_0 + 1,\ |A_1| = k_1 + 1 \Rightarrow$
  $\Rightarrow \dim N(R) = k_0 + k_1 - 1$, $\binom{k_0+k_1}{k_0}$ vertices, $k_0 k_1 + 3$ facets.

- [Sturmfels'94] Multivariate case / up to 3 dimensions



The only resultant polytopes up to dimension 3

One step beyond: 4-dimensional N(R)

- f-vector of face cardinalities: of vertices, edges, ridges, facets.
- Some f-vectors (some by 3 generic triangles):

(5, 10, 10, 5): 4-simplex
(6, 15, 18, 9): Birkhoff
(8, 20, 21, 9)
(9, 22, 21, 8)
...
(10, 26, 25, 9): Sylvester, $k_i \in \{2, 3\}$
...
(17, 50, 50, 17)
(18, 51, 48, 15)
(18, 51, 49, 16)
(18, 52, 50, 16)
(18, 52, 51, 17)
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(18, 53, 53, 18)
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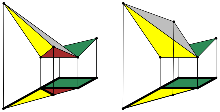
(19, 57, 57, 19)
(20, 58, 54, 16)
(20, 59, 57, 18)
(20, 60, 60, 20)
(21, 62, 60, 19)
(21, 63, 63, 21)
(22, 66, 66, 22)

Computation of resultant polytopes



- `respol` software [Emiris-F-Konaxis-Peñaranda]
  - offers lower bounds on f-vectors
  - C++, `http://sourceforge.net/projects/respol`
  - 5-, 6- and 7-d polytopes with 35K, 23K and 500 vertices, resp., within 2hrs; the secondary polytopes are intractable.
- Alternative algorithm employs tropical geometry (`GFan` library) [Jensen-Yu]

Main result

Theorem

Given essential family $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$, with $N(R)$ of dimension 4, $N(R)$ is (a degeneration of) any of the following polytopes:

(i) *if all $|A_i| = 2$, except for one with cardinality 5, the 4-simplex with f-vector $(5, 10, 10, 5)$.*

(ii) *if all $|A_i| = 2$, except for two with cardinalities 3 and 4, a polytope with f-vector $(10, 26, 25, 9)$.*

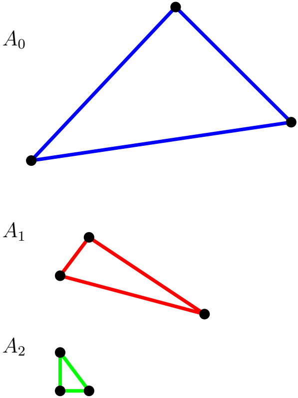
(iii) *if all $|A_i| = 2$, except for three with cardinality 3, a polytope with maximal face numbers $\tilde{f}_3 = 22$, $\tilde{f}_2 = 66$, $\tilde{f}_1 = \tilde{f}_0 + 44$, and $22 \leq \tilde{f}_0 \leq 28$.*

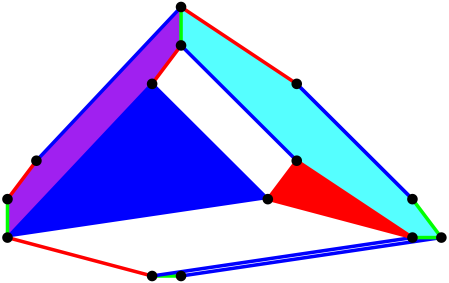
- Degenarations can only decrease the number of faces.
- Focus on new case (iii): reduces to $n = 2$ and $|A_i| = 3$.
- Previous upper bound for vertices yields 6608 [Sturmfels'94].

Subdivisions of the Minkowski sum

A subdivision S of $A_0 + A_1 + \cdots + A_n \subset \mathbb{Z}^n$:

- is mixed when its cells have (unique) expressions as Minkowski sums of subsets of the A_i, and the cells intersect properly.



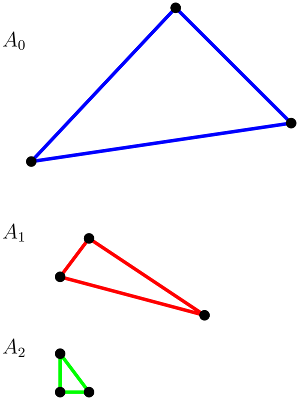


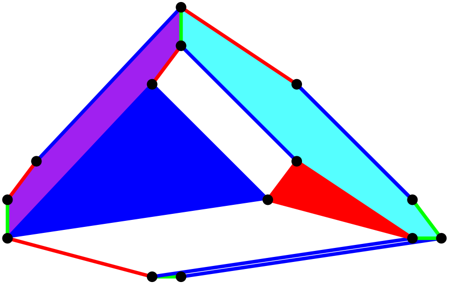
mixed subdivision S of $A_0 + A_1 + A_2$

Subdivisions of the Minkowski sum

A subdivision S of $A_0 + A_1 + \cdots + A_n \subset \mathbb{Z}^n$:

- is fine (or tight) if, for all cells $\sigma = \sum_{i=0}^{n} F_i$, it holds $\dim \sigma = \sum_{i=0}^{n} \dim F_i$; otherwise, it is coarse.



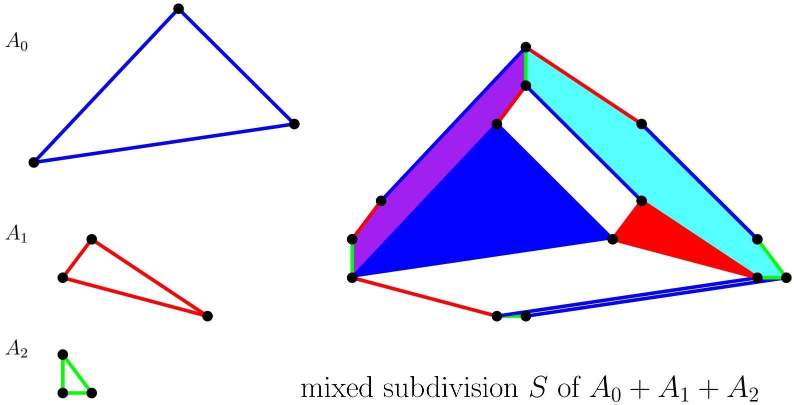


mixed subdivision S of $A_0 + A_1 + A_2$

Subdivisions of the Minkowski sum

A subdivision S of $A_0 + A_1 + \cdots + A_n \subset \mathbb{Z}^n$:

- is regular if obtained by projecting the lower hull of the Minkowski sum of the $\widehat{A}_0, \ldots, \widehat{A}_n \subset \mathbb{Z}^{n+1}$, defined by some (typically linear) lifting function $w : \mathbb{R}^n \to \mathbb{R}^{n+1}$;



mixed subdivision S of $A_0 + A_1 + A_2$

Mapping subdivisions to $N(R)$ faces

Proposition (GKZ,Sturmfels)

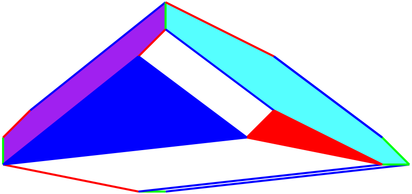
Consider the regular mixed subdivision S *of* $A_0 + A_1 + \cdots + A_n$*, obtained by a lifting defined by* $w \in \mathbb{R}^m$*. Then,* S *defines a face of* $N(R)$ *which has* w *as outer normal, equal to the Newton polytope of*

$$\prod_{\sigma \in S} R(f_0|_\sigma, \ldots, f_n|_\sigma)^{d_\sigma},$$

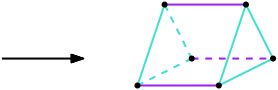
i.e. the Minkowski sum of the resultant polytopes of subsystems $\{f_0|_\sigma, \ldots, f_n|_\sigma\}$ *correspoding to cells* $\sigma \in S$*, where* d_σ *is the normalized volume of* σ*.*

Resultant facets

- white, blue, red cells: contain point summand
- purple cell: 2 subdivisions $\to N(R)$ segment
- turquoise cell: 3 subdivisions $\to N(R)$ triangle



subd. S of $A_0 + A_1 + A_2$



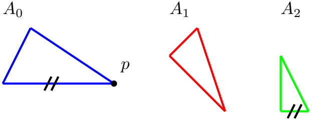
Mink. sum of $N(R)$ triangle and $N(R)$ segment

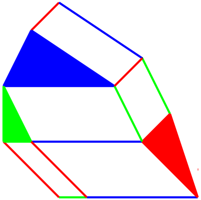
Resultant vertices are obtained from fine regular mixed subdivisions

Genericity maximizes complexity

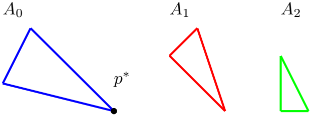
Theorem

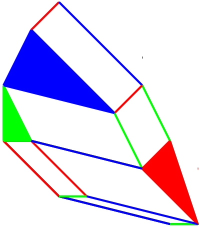
The number of $N(R)$ faces for 3 triangles is maximized for generic triangles, namely 2-d, without parallel edges.





$N(R)$ f-vector: $(14, 38, 36, 12)$



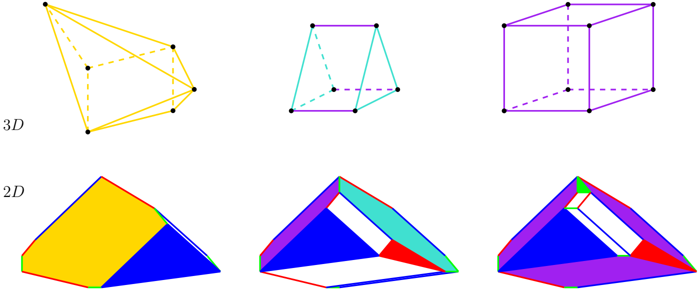


$N(R^*)$ f-vector: $(18, 52, 50, 16)$

Possible facets

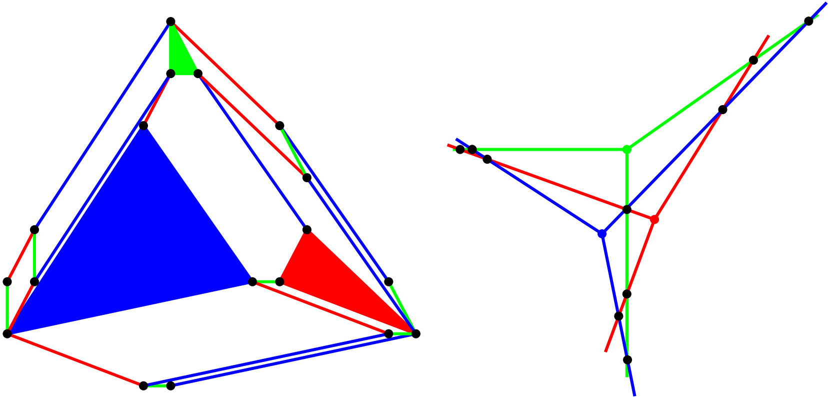
Lemma

- *resultant facet:* 3*-d N(R): octagon in* S*,*
- *prism:* 2*-d N(R) (triangle)* $+$ 1*-d N(R): heptagon and hexagon,*
- *zonotope:* 1*-d N(R)* $+$ 1*-d N(R)* $+$ 1*-d N(R): 3 hexagons.*



Counting facets

A technical tool: duality of mixed subdivisions.

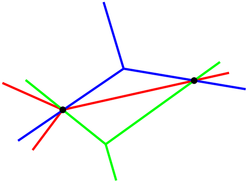


Counting facets

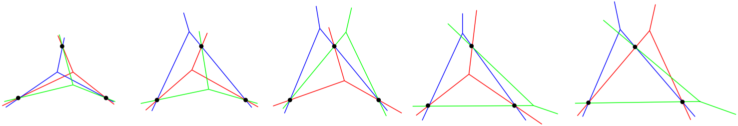
Lemma

Maximal face numbers are as follows:

- *9 resultant facets: 9 octagons with* $|A_i| = 3, 3, 2$.
- *9 prisms: 9 hexagon-heptagon pairs: unique subdivision if common edge picked, i.e., common dual ray fixed.*



- *4 zonotopes: 4 triplets of tri-chromatic points.*



Face numbers

Theorem

The maximal number of facets and ridges is $\tilde{f}_3 = 22$ *and* $\tilde{f}_2 = 66$*, resp. Moreover,* $\tilde{f}_1 = \tilde{f}_0 + 44$*, and* $22 \leq \tilde{f}_0 \leq 28$*. The lower bounds are tight.*

Proof

- Count 36 triangular and 30 parallelogram ridges.
- [Kalai'87] If $f_2^i = \#$ two-faces which are i-gons,

$$f_1 + \sum_{i \geq 4} (i-3) f_2^i \ \geq \ d f_0 - \binom{d+1}{2},$$

hence $f_1 + 30 \geq 4 f_0 - 10$, then apply Euler's equation.

Questions

- Is the maximum f-vector of a 4d-resultant polytope $(22, 66, 66, 22)$?
- Explain symmetry of maximal f-vectors for non self-dual polytopes
- Compute the subdivision with max/min cells

Conjecture

Let $\tilde{f}_0(d) = \max$ #vertices of a d-dimensional $N(R)$:

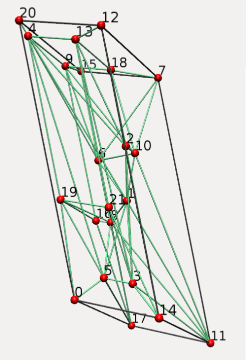
$$\tilde{f}_0(d) \;\leq\; (3d-3) \cdot \sum_{\|T\|=d-1} \; \prod_{i \in T} \tilde{f}_0(i),$$

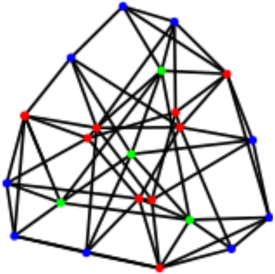
where T is any multiset with elements in $\{1, \ldots, d-1\}$, $\|T\| = \sum_{i \in T} i$.

Gives $\tilde{f}_0(3) \leq 42$ (instead of 6), $\tilde{f}_0(4) \leq 180$ (instead of 28).

Bound in d [Sturmfels'94] is $(3d-3)^{2d^2} \Rightarrow \tilde{f}_0(5) \leq 12^{50}$; the conjecture yields 924

Schlegel diagram and facet graph [M. Joswig] of maximal polytope





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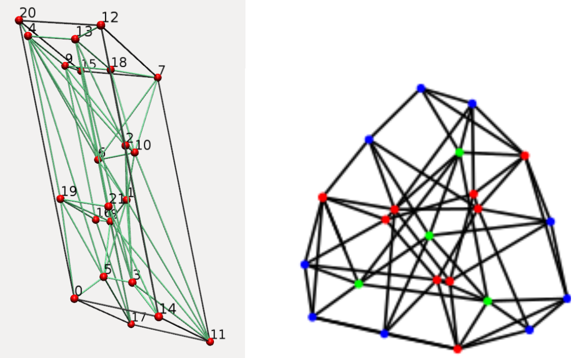

Thank you!