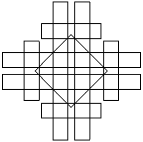
An output-sensitive algorithm for computing (projections of) resultant polytopes

Vissarion Fisikopoulos

Joint work with I.Z. Emiris, C. Konaxis and L. Peñaranda

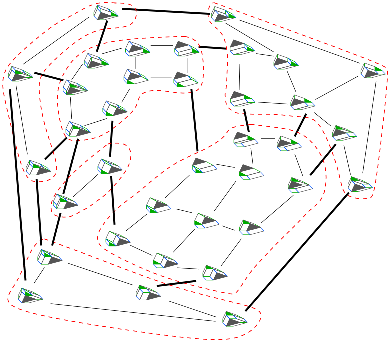
Department of Informatics, University of Athens

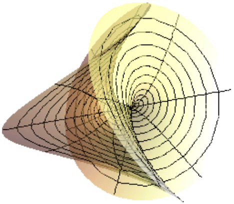


SoCG, Chapel Hill, NC, USA, 18.Jun.2012

An interesting class of polytopes: resultant polytopes

- Geometry: Minkowski summands of secondary polytopes, equivalence classes of secondary vertices, generalization of Birkhoff polytopes
- Motivation: useful to express the solvability of polynomial systems
- Applications: discriminant and resultant computation, implicitization of parametric hypersurfaces





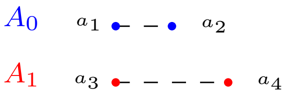
Enneper's Minimal Surface

Existing work

- Theory of resultants, secondary polytopes, Cayley trick [GKZ '94]
- TOPCOM [Rambau '02] computes all vertices of secondary polytope.
- [Michiels & Verschelde DCG'99] define and enumerate coarse equivalence classes of secondary polytope vertices.
- [Michiels & Cools DCG'00] describe a decomposition of $\Sigma(\mathcal{A})$ in Minkoski summands, including $N(\mathcal{R})$.
- Tropical geometry [Sturmfels-Yu '08] leads to algorithms for the resultant polytope (`GFan` library) [Jensen-Yu '11] and the discriminant polytope (`TropLi` software) [Rincón '12].

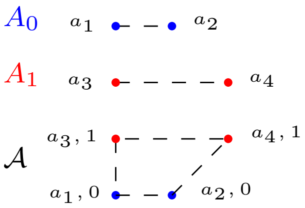
What is a resultant polytope?

- Given $n+1$ point sets $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$



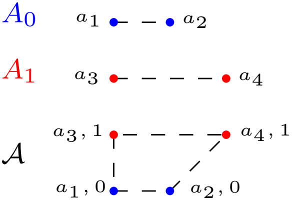
What is a resultant polytope?

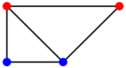
- Given $n+1$ point sets $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$
- $\mathcal{A} = \bigcup_{i=0}^{n} (A_i \times \{e_i\}) \subset \mathbb{Z}^{2n}$ where $e_i = (0, \ldots, 1, \ldots, 0) \subset \mathbb{Z}^n$



What is a resultant polytope?

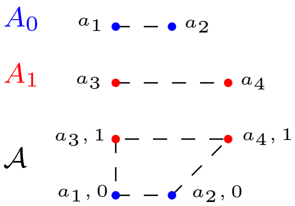
- Given $n+1$ point sets $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$
- $\mathcal{A} = \bigcup_{i=0}^{n} (A_i \times \{e_i\}) \subset \mathbb{Z}^{2n}$ where $e_i = (0, \ldots, 1, \ldots, 0) \subset \mathbb{Z}^n$
- Given T a triangulation of $conv(\mathcal{A})$, a cell is a-mixed if it is the Minkowski sum of n 1-dimensional segments from A_j, $j \neq i$, and some vertex $a \in A_i$.

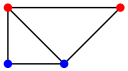




What is a resultant polytope?

- Given $n+1$ point sets $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$
- $\mathcal{A} = \bigcup_{i=0}^{n} (A_i \times \{e_i\}) \subset \mathbb{Z}^{2n}$ where $e_i = (0, \ldots, 1, \ldots, 0) \subset \mathbb{Z}^n$
- Given T a triangulation of $conv(\mathcal{A})$, a cell is a-mixed if it is the Minkowski sum of n 1-dimensional segments from A_j, $j \neq i$, and some vertex $a \in A_i$.
- $\rho_T(a) = \sum_{\substack{a-\text{mixed} \\ \sigma \in T : a \in \sigma}} \text{vol}(\sigma) \ \in \mathbb{N}, \quad a \in \mathcal{A}$

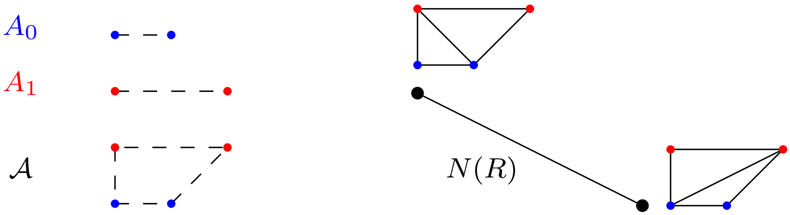




$\rho_T = (0, 2, 1, 0)$

What is a resultant polytope?

- Given $n+1$ point sets $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$
- $\mathcal{A} = \bigcup_{i=0}^{n} (A_i \times \{e_i\}) \subset \mathbb{Z}^{2n}$ where $e_i = (0, \ldots, 1, \ldots, 0) \subset \mathbb{Z}^n$
- Given T a triangulation of $conv(\mathcal{A})$, a cell is a-mixed if it is the Minkowski sum of n 1-dimensional segments from A_j, $j \neq i$, and some vertex $a \in A_i$.
- $\rho_T(a) = \sum_{\substack{a-\text{mixed} \\ \sigma \in T : a \in \sigma}} \text{vol}(\sigma) \ \in \mathbb{N}, \quad a \in \mathcal{A}$
- Resultant polytope $N(R) = conv(\rho_T : T \ \text{ triang. of } conv(\mathcal{A}))$



Connection with Algebra

- The Newton polytope of f, $N(f)$, is the convex hull of the set of exponents of its monomials with non-zero coefficient.
- The resultant R is the polynomial in the coefficients of a system of polynomials which is zero iff the system has a common solution.

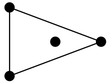
A_0

$$f_0(x) = ax^2 + b$$

A_1

$$f_1(x) = cx^2 + dx + e$$

$N(R)$



$$R(a,b,c,d,e) = ad^2b + c^2b^2 - 2caeb + a^2e^2$$

Connection with Algebra

- The Newton polytope of f, $N(f)$, is the convex hull of the set of exponents of its monomials with non-zero coefficient.
- The resultant R is the polynomial in the coefficients of a system of polynomials which is zero iff the system has a common solution.

A_0 

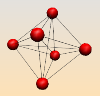
$$f_0(x,y) = ax + by + c$$

A_1 

$$f_1(x,y) = dx + ey + f$$

A_2 

$$f_2(x,y) = gx + hy + i$$

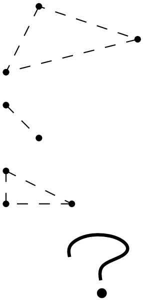
$N(R)$ 

4-dimensional Birkhoff polytope

$$R(a,b,c,d,e,f,g,h,i) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Connection with Algebra

- The Newton polytope of f, $N(f)$, is the convex hull of the set of exponents of its monomials with non-zero coefficient.
- The resultant R is the polynomial in the coefficients of a system of polynomials which is zero iff the system has a common solution.



A_0 $$f_0(x,y) = axy^2 + x^4y + c$$

A_1 $$f_1(x,y) = dx + ey$$

A_2 $$f_2(x,y) = gx^2 + hy + i$$

$N(R)$

NP-hard to compute the resultant
in the general case

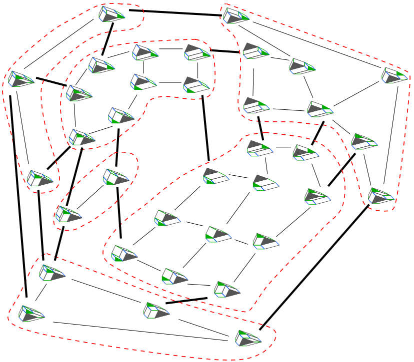
The idea of the algorithm

Input: $\mathcal{A} \in \mathbb{Z}^{2n}$ defined by $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$

Simplistic method:

- compute the secondary polytope $\Sigma(\mathcal{A})$
- many-to-one relation between vertices of $\Sigma(\mathcal{A})$ and $N(R)$ vertices

Cannot enumerate 1 representative per class by walking on secondary edges

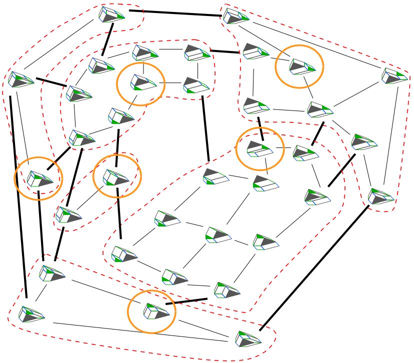


The idea of the algorithm

Input: $\mathcal{A} \in \mathbb{Z}^{2n}$ defined by $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$

New Algorithm:

- Vertex oracle: given a direction vector compute a vertex of $N(R)$
- Output sensitive: computes only one triangulation of $\mathcal{A}$ per $N(R)$ vertex + one per $N(R)$ facet
- Computes projections of $N(R)$ or $\Sigma(\mathcal{A})$

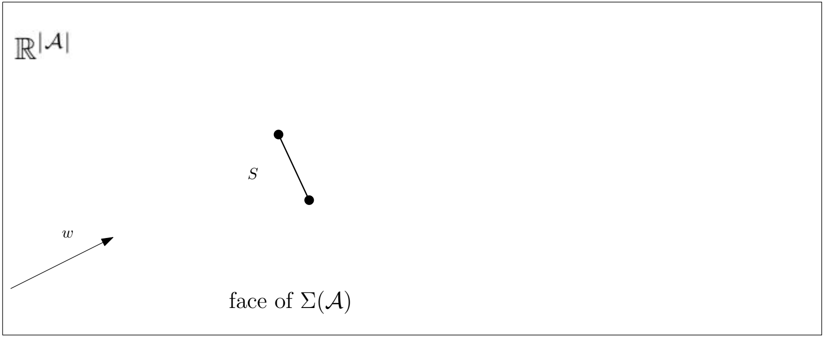


The Oracle

Input: $\mathcal{A} \subset \mathbb{Z}^{2n}$, direction $w \in (\mathbb{R}^{|\mathcal{A}|})^{\times}$

Output: vertex $\in N(R)$, extremal wrt w

1. use w as a lifting to construct regular subdivision S of $\mathcal{A}$

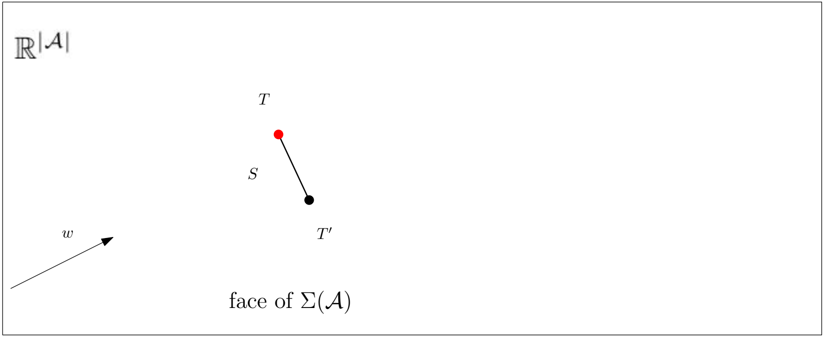


The Oracle

Input: $\mathcal{A} \subset \mathbb{Z}^{2n}$, direction $w \in (\mathbb{R}^{|\mathcal{A}|})^\times$

Output: vertex $\in N(R)$, extremal wrt w

1. use w as a lifting to construct regular subdivision S of $\mathcal{A}$
2. refine S into triangulation T of $\mathcal{A}$

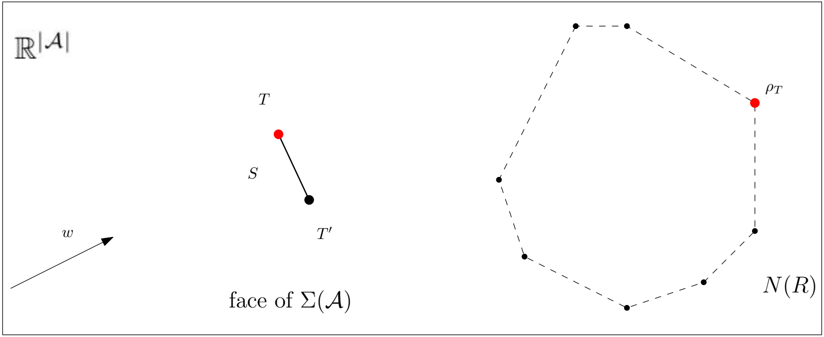


The Oracle

Input: $\mathcal{A} \subset \mathbb{Z}^{2n}$, direction $w \in (\mathbb{R}^{|\mathcal{A}|})^{\times}$

Output: vertex $\in N(R)$, extremal wrt w

1. use w as a lifting to construct regular subdivision S of $\mathcal{A}$
2. refine S into triangulation T of $\mathcal{A}$
3. return $\rho_T \in \mathbb{N}^{|\mathcal{A}|}$



The Oracle

Input: $\mathcal{A} \subset \mathbb{Z}^{2n}$, direction $w \in (\mathbb{R}^{|\mathcal{A}|})^{\times}$

Output: vertex $\in N(R)$, extremal wrt w

1. use w as a lifting to construct regular subdivision S of $\mathcal{A}$
2. refine S into triangulation T of $\mathcal{A}$
3. return $\rho_T \in \mathbb{N}^{|\mathcal{A}|}$

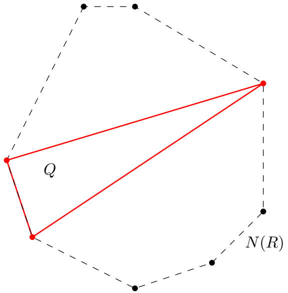
Oracle property: its output is a vertex of the target polytope (Lem. 5).

Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step



initialization:

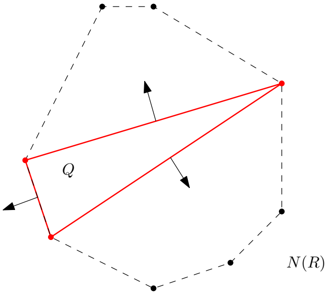
- $Q \subset N(R)$
- $\dim(Q) = \dim(N(R))$

Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal



2 kinds of hyperplanes of Q_H:

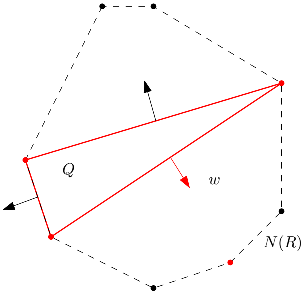
- legal if it supports facet $\subset N(R)$
- illegal otherwise

Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$

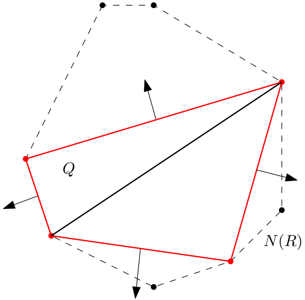


Extending an illegal facet

Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal

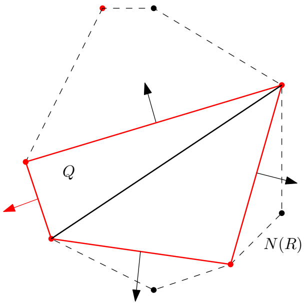


Extending an illegal facet

Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal

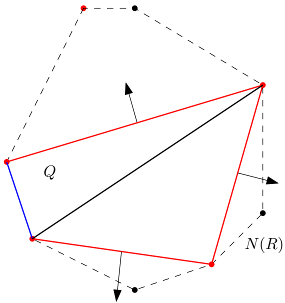


Validating a legal facet

Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



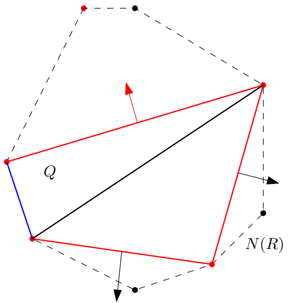
Validating a legal facet

Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

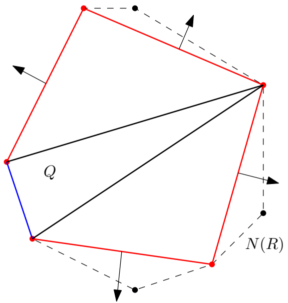
1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal

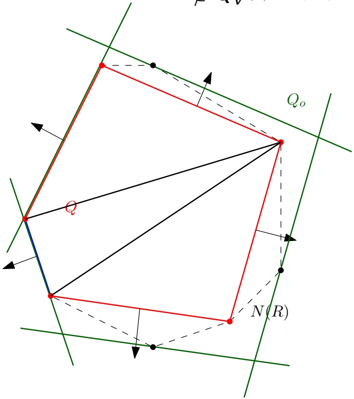


At any step, Q is an inner approximation . . .

Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



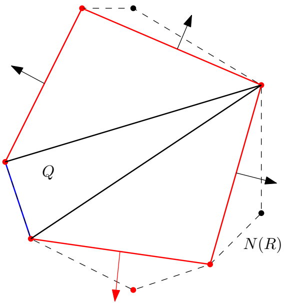
At any step, Q is an inner approximation ... from which we can compute an outer approximation Q_o.

Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal

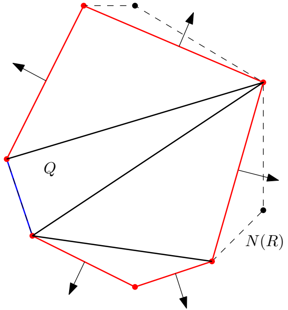


Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal

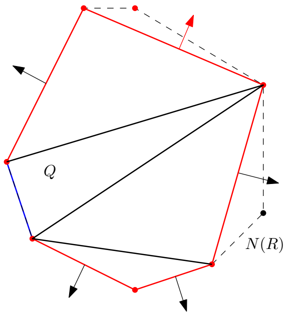


Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

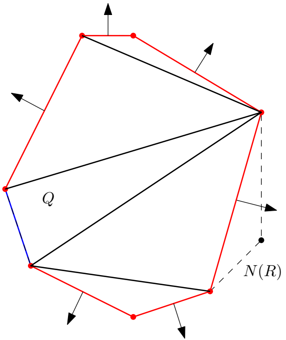
1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal

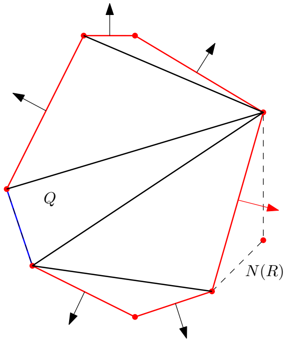


Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

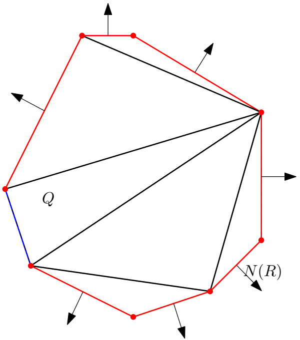
1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal

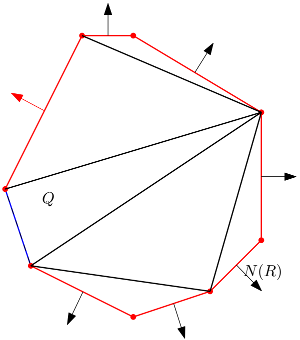


Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

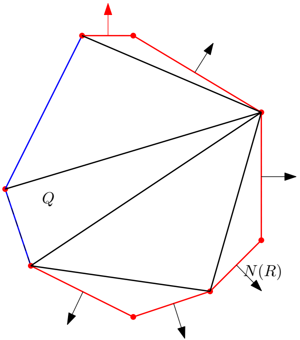
1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

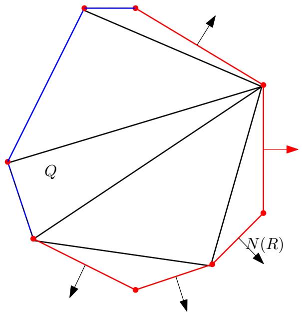
1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

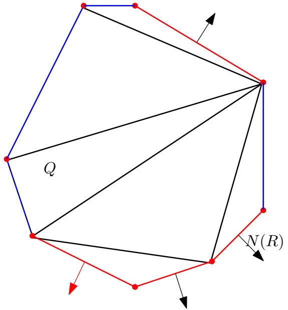
1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal

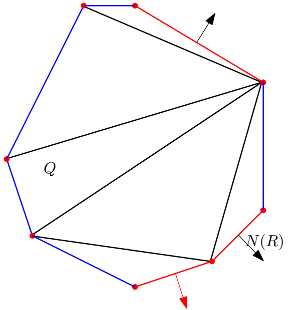


Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal

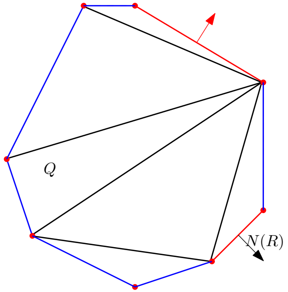


Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

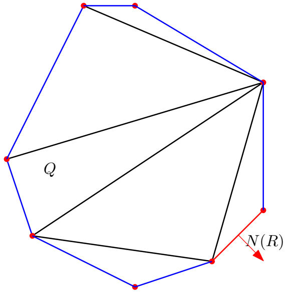
1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

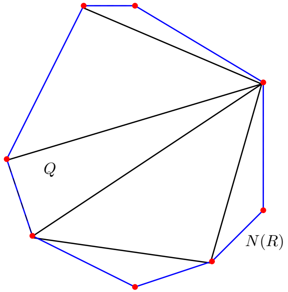
1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



Incremental Algorithm

Input: $\mathcal{A}$
Output: H-rep. Q_H, V-rep. Q_V of $Q = N(R)$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w `do`
   - call oracle for w and compute v, $Q_V \leftarrow Q_V \cup \{v\}$
   - `if` $v \notin Q_V \cap H$ `then` $Q_H \leftarrow \mathrm{CH}(Q_V \cup \{v\})$ `else` H is legal



Complexity

Theorem

We compute the Vertex- and Halfspace-representations of $N(R)$, as well as a triangulation T of $N(R)$, in

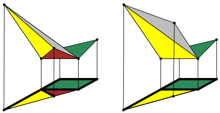
$$O^*(\, m^5\, |vtx(N(R))| \cdot |T|^2),$$

where $m = \dim N(R)$*, and* $|T|$ *the number of full-dim faces of* T*.*

Elements of proof

- Computation is done in dimension $m = |\mathcal{A}| - 2n + 1$.
- At most $\leq \mathrm{vtx}(N(R)) + \mathrm{fct}(N(R))$ oracle calls (Lem. 9).
- Beneath-and-Beyond algorithm for converting V-rep. to H-rep [Joswig '02].

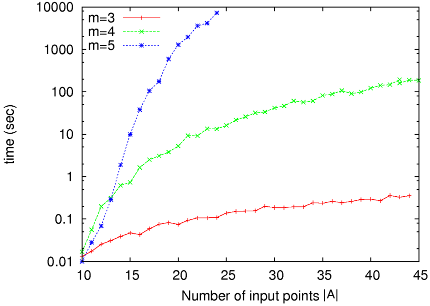
ResPol package

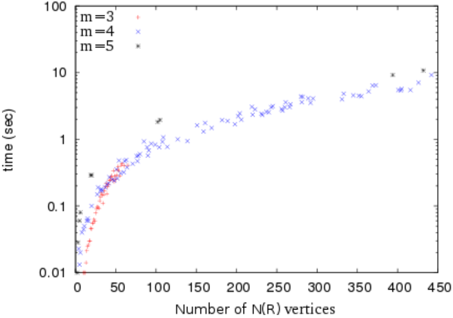


- C++
- CGAL, `triangulation` [Boissonnat,Devillers,Hornus]
  `extreme_points_d` [Gärtner] (preprocessing step)
- Hashing of determinantal predicates: optimizing sequences of similar determinants
- `http://sourceforge.net/projects/respol`
- Applications of ResPol on I.Emiris talk this afternoon (CGAL, an Open Gate to Computational Geometry!)

Output-sensitivity

- oracle calls $\leq \text{vtx}(N(R)) + \text{fct}(N(R))$
- output vertices bound polynomially the output triangulation size
- subexponential runtime wrt to input points (L), output vertices (R)

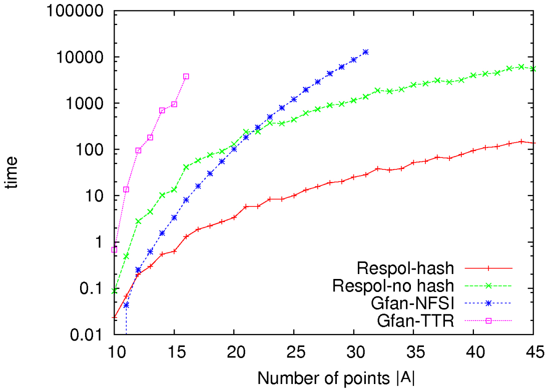




Hashing and `Gfan`

- *hashing determinants* speeds ≤ 10-100x when $dim(N(R)) = 3, 4$
- faster than `Gfan` [Yu-Jensen'11] for $dimN(R) \leq 6$, else competitive

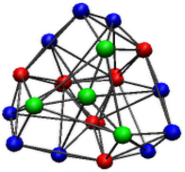
$dim(N(R)) = 4$:



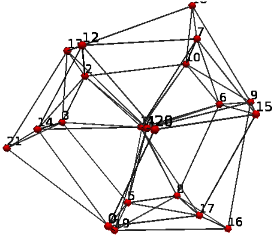
Ongoing and future work

- approximate resultant polytopes ($dim(N(R)) \geq 7$) using approximate volume computation
- combinatorial characterization of 4-dimensional resultant polytopes
- computation of discriminant polytopes

More on I.Emiris talk this afternoon (CGAL, an Open Gate to Computational Geometry!)

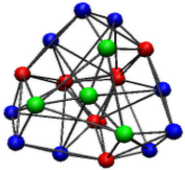


(figure courtesy of M.Joswig)

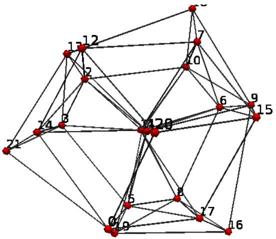


Facet and vertex graph of the largest 4-dimensional resultant polytope

Ongoing and future work



(figure courtesy of M.Joswig)

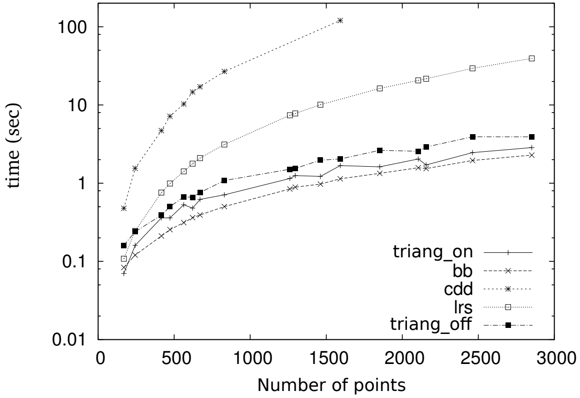


Facet and vertex graph of the largest 4-dimensional resultant polytope

Thank You !

Convex hull implementations

- From V- to H-rep. of $N(R)$.
- `triangulation (on/off-line)`, polymake `beneath-beyond`, `cdd`, `lrs`



$dim(N(R)) = 4$