Polyhedral computations in computational algebraic geometry and optimization

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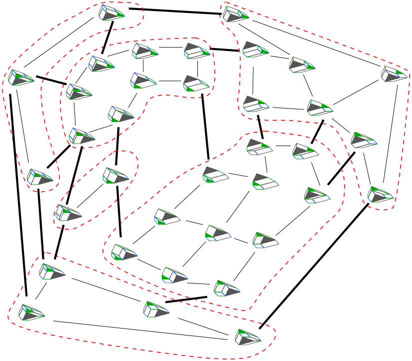
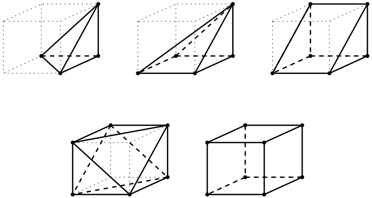
LSE, Lunchtime Seminar, 15 May 2015

Outline of the talk

Polytopes defined by oracles: computation & combinatorics

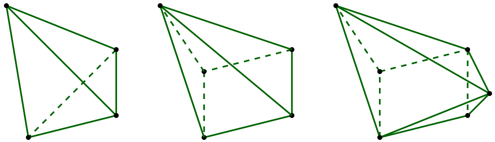
Enumeration of 2-level polytopes





Motivation: resultant polytopes

- Algebra:
  - complexity of resultant polynomial
- Geometry:
  - generalize Birkhoff polytopes
  - faces are Minkowski sums of resultant polytopes
  - vertices → triangulations ↔ subdivisions of Mink. sums
- Applications:
  - support computation → interpolate implicit equation of parametric hypersurface
  - compute resultants, discriminants



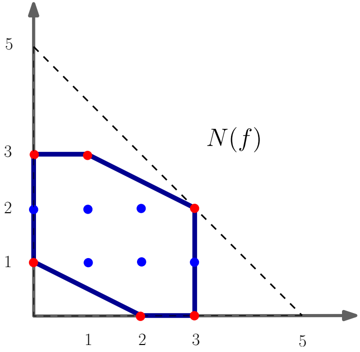
Examples of resultant polytopes

Newton polytope

Definition

Given polynomial $f \in K[x_1, \dots, x_n]$ the Newton polytope $N(f)$ of f is the convex hull of the support, i.e. exponent vectors of monomials with non-zero coefficient.

$$f(x_1, x_2) = 8x_2 + x_1x_2 - 24x_2^2 - 16x_1^2 + 220x_1^2x_2 - 34x_1x_2^2 - 84x_1^3x_2 + 6x_1^2x_2^2 - 8x_1x_2^3 + 8x_1^3x_2^2 + 8x_1^3 + 18x_2^3$$



Polytopes and Algebra

Definition

Given are polynomials $f_0, f_1, \dots, f_n \in K[x_1, \dots, x_n]$, s.t. the supports define an essential family $A_0, A_1, \dots, A_n \subset \mathbb{Z}^n$, i.e. the A_i generate $\mathbb{Z}^n$ and any k-subset generates a sublattice of dimension $\geq k$.

The system's (sparse) resultant R is the polynomial in the system's coefficients, defined up to sign, which vanishes iff the polynomials have a common root in the corresponding toric variety X: $(\overline{K}^*)^n \subset X$.

The resultant polytope $N(R)$ is the Newton polytope of R.

$f_0(x) = ax^2 + b$

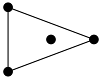
A_0

$f_1(x) = cx^2 + dx + e$

A_1

$R(a, b, c, d, e) = ad^2b + c^2b^2 - 2caeb + a^2e^2$

$N(R)$



Birkhoff polytope

Linear polynomials

A_0

$$f_0(x,y) = ax + by + c$$

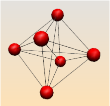
A_1

$$f_1(x,y) = dx + ey + f$$

A_2

$$f_2(x,y) = gx + hy + i$$

$N(R)$



4-dimensional Birkhoff polytope

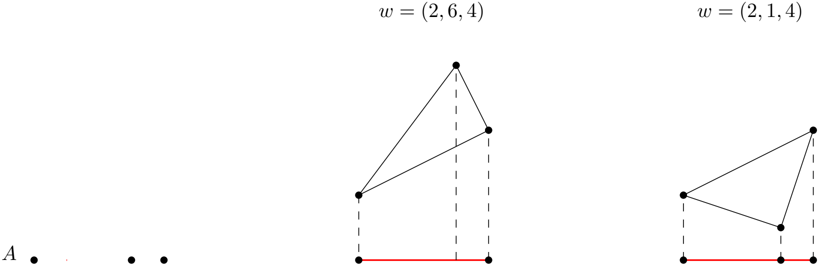
$$R(a,b,c,d,e,f,g,h,i) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Existing work

- Resultants, secondary polytopes, Cayley trick [GKZ '94]
- TOPCOM [Rambau '02] computes all vertices of secondary polytope.
- [Michiels & Verschelde DCG'99] coarse equivalence classes of secondary polytope vertices.
- [Michiels & Cools DCG'00] decomposition of $\Sigma(\mathcal{A})$ in Minkoski summands, including $N(\mathcal{R})$.
- Tropical geometry [Sturmfels-Yu '08]: algorithms for resultant polytope (`GFan` library) [Jensen-Yu '11] and discriminant polytope (`TropLi` software) [Rincn '12].

Regularity

Regular subdivision of $A \subset \mathbb{R}^d$ are obtained by projecting the lower (or upper) hull of A lifted to $\mathbb{R}^{d+1}$ via a lifting function $w \in (\mathbb{R}^{|\mathcal{A}|})^\times$.

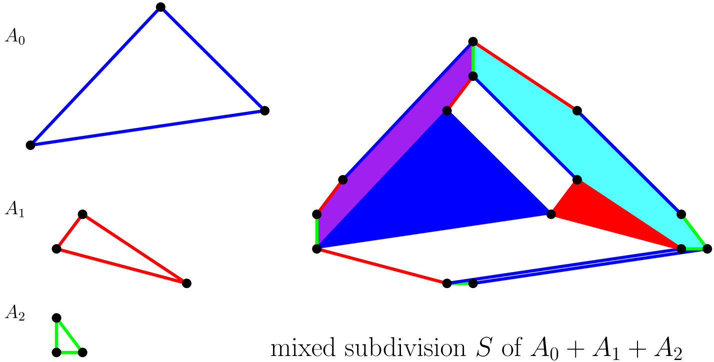


Resultant polytope vertices and mixed subdivisions

A subdivision S of $A_0 + A_1 + \cdots + A_n$ is

- mixed when its cells have expressions as Minkowski sums of convex hulls of point subsets in A_i's,
- fine when each cell has dimension equal to the sum of its summands dimensions.

Example



mixed subdivision S of $A_0 + A_1 + A_2$

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Theorem [GKZ '94, Sturmfels '94]

- many-to-one relation between regular fine mixed subdivisions and $N(R)$ vertices
- one-to-one relation between regular fine mixed subdivisions and secondary polytope $\Sigma(\mathcal{A})$ vertices

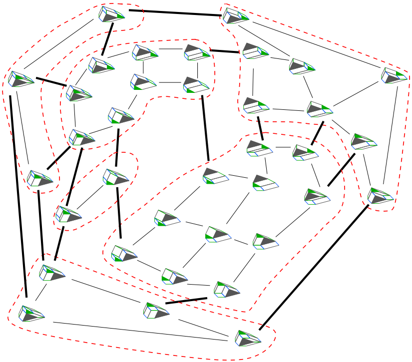
The idea of the algorithm for N(R)

Input: $\mathcal{A} \in \mathbb{Z}^{2n}$ defined by $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$

Simplistic method:

- compute the secondary polytope $\Sigma(\mathcal{A})$
- many-to-one relation between vertices of $\Sigma(\mathcal{A})$ and $N(R)$

Cannot enum 1 representative/class by walking on secondary edges

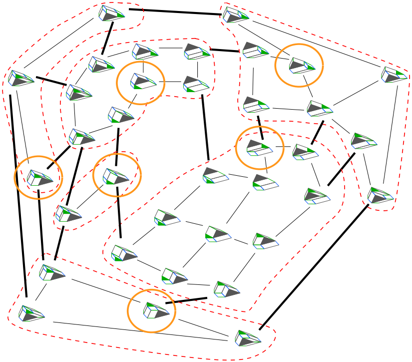


The idea of the algorithm for N(R)

Input: $\mathcal{A} \in \mathbb{Z}^{2n}$ defined by $A_0, A_1, \ldots, A_n \subset \mathbb{Z}^n$

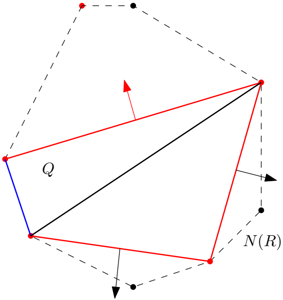
New Algorithm:

- Vertex oracle: given direction vector compute a vertex of $N(R)$ by computing a subdivision using the direction as lifting
- Output sensitive: computes only one subdivision of $\mathcal{A}$ per $N(R)$ vertex + one per $N(R)$ facet
- Computes projections of $N(R)$ or $\Sigma(\mathcal{A})$



Incremental algorithm for $N(R)$

- first: compute conv.hull of $d+1$ aff. indep. vertices of $N(R)$
- step: call the oracle with outer normal vector of a halfspace
  $\rightarrow$ either validate this halfspace
  $\rightarrow$ or add a new vertex to the convex hull



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Theorem (Emiris,F,Konaxis,Penaranda)

Given $P \subseteq \mathbb{R}^d$, H-, V-repr. & triang. T of $N(R)$ can be computed in

$$O(d^5 n s^2) \text{ arithmetic operations } + O(n+m) \text{ oracle calls}$$

s is the number of cells of T.

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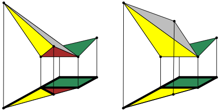
BUT: s can be $O\left(n^{\lfloor d/2 \rfloor}\right)$

ResPol package

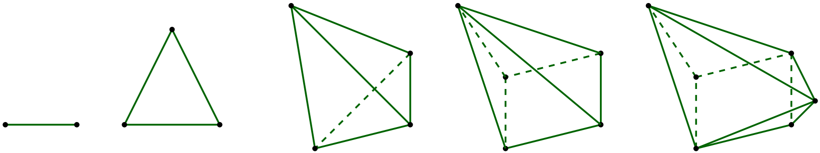


- Towards high-dimensional CGAL (Computational Geometry Algorithms Library)
- Hashing of determinantal predicates scheme: optimizing sequences of similar determinants (x100 speed-up)
- Computes 5-, 6- and 7-dimensional polytopes with 35K, 23K and 500 vertices, respectively, within 2hrs
- Computes polytopes of many important surface equations encountered in geometric modeling in < 1sec, whereas the corresponding secondary polytopes are intractable
- `http://sourceforge.net/projects/respol`

Combinatorics of resultant polytopes

- [GKZ'90] Univariate case, general-dimensional $N(R)$:
  The A_i are multisets from $\mathbb{Z}$: $|A_0| = k_0 + 1,\ |A_1| = k_1 + 1 \Rightarrow$
  $\Rightarrow \dim N(R) = k_0 + k_1 - 1$, $\binom{k_0+k_1}{k_0}$ vertices, $k_0 k_1 + 3$ facets.

- [Sturmfels'94] Multivariate case / up to 3 dimensions



The only resultant polytopes up to dimension 3

One step beyond: 4-dimensional N(R)

- f-vector of face cardinalities: of vertices, edges, ridges, facets.
- Some f-vectors (generic input):

(5, 10, 10, 5): 4-simplex
(6, 15, 18, 9): Birkhoff
(8, 20, 21, 9)
(9, 22, 21, 8)
. . .
(10, 26, 25, 9): Sylvester, $k_i \in \{2, 3\}$
. . .
(17, 50, 50, 17)
(18, 51, 48, 15)
(18, 51, 49, 16)
(18, 52, 50, 16)
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(19, 57, 57, 19)
(20, 58, 54, 16)
(20, 59, 57, 18)
(20, 60, 60, 20)
(21, 62, 60, 19)
(21, 63, 63, 21)
(22, 66, 66, 22)

Combinatorics of 4-dim resultant polytopes

Theorem (Dickenstein,Emiris,F)

Given essential family $A_0, A_1, \dots, A_n \subset \mathbb{Z}^n$*, with* $N(R)$ *of dimension* 4*,* $N(R)$ *is (a degeneration of) any of the following polytopes:*

(i) $|A_i| : 2 \dots 2, 5$, $N(R)$ *is the* 4*-simplex,* f*-vector* $(5, 10, 10, 5)$.

(ii) $|A_i| : 2 \dots 2, 3, 4$, $N(R)$ f*-vector* $(10, 26, 25, 9)$.

(iii) $|A_i| : 2 \dots 2, 3, 3, 3$, $N(R)$ *has maximal face numbers* $\tilde{f}_3 = 22$, $\tilde{f}_2 = 66$, $\tilde{f}_1 = \tilde{f}_0 + 44$, *and* $22 \leq \tilde{f}_0 \leq 28$.

- Degenarations can only decrease the number of faces.
- Previous upper bound for vertices yields 6608 [Sturmfels'94].
- Focus on new case (iii): reduces to $n = 2$ and $|A_0| = |A_1| = |A_2| = 3$

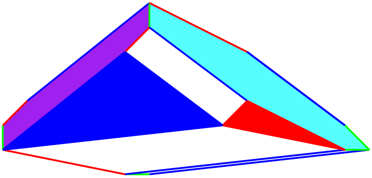
Mixed subdivisions and $N(R)$ faces (I)

Proposition (GKZ,Sturmfels)

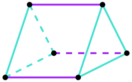
Consider the regular mixed subdivision S of $A_0 + A_1 + \cdots + A_n$, obtained by a lifting defined by $w \in \mathbb{R}^m$. Then, S defines a face of $N(R)$ which has w as outer normal, equal to the Newton polytope of

$$\prod_{\sigma \in S} R(f_0|_\sigma, \ldots, f_n|_\sigma)^{d_\sigma},$$

i.e. the Minkowski sum of the resultant polytopes of subsystems $\{f_0|_\sigma, \ldots, f_n|_\sigma\}$ correspods to cells $\sigma \in S$, where d_σ is the normalized volume of σ.



subd. S of $A_0 + A_1 + A_2$

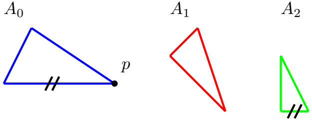


Mink. sum of $N(R)$ triangle and $N(R)$ segment

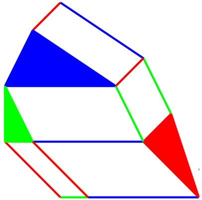
Genericity maximizes complexity (II)

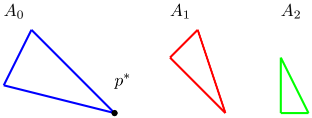
Theorem

The number of $N(R)$ faces for 3 triangles is maximized for generic triangles, namely 2-d, without parallel edges.

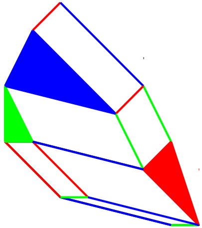


$N(R)$ f-vector: $(14, 38, 36, 12)$





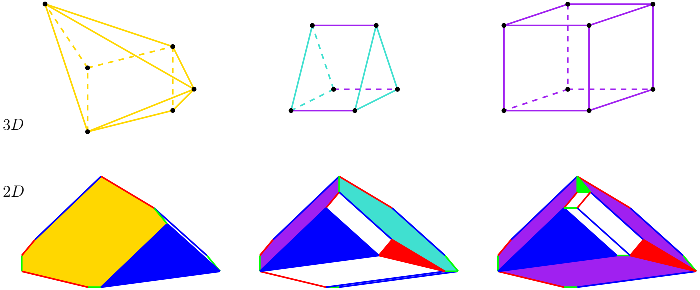
$N(R^*)$ f-vector: $(18, 52, 50, 16)$



Possible facets

Lemma

- *resultant facet: 3-d N(R): octagon in S,*
- *prism: 2-d N(R) (triangle) $+$ 1-d N(R): heptagon and hexagon,*
- *zonotope: 1-d N(R) $+$ 1-d N(R) $+$ 1-d N(R): 3 hexagons.*

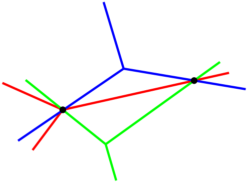


Counting facets and duality of mixed subdivisions (III)

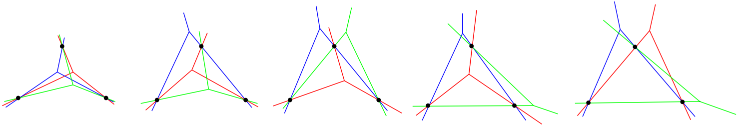
Lemma

Maximal face numbers are as follows:

- *9 resultant facets: 9 octagons with* $|A_i| = 3, 3, 2$.
- *9 prisms: 9 hexagon-heptagon pairs: unique subdivision if common edge picked, i.e., common dual ray fixed.*



- *4 zonotopes: 4 triplets of tri-chromatic points.*



Extensions - Open problems

- Algorithmic
    - Total polynomial algorithms for CH (edge-directions [Emiris-F-Gartner])
    - Volume computation (randomized implementation [Emiris-F])
    - Lattice points enumeration
- Combinatorial
    - The maximum f-vector of a 4d N(R) is $(22, 66, 66, 22)$
    - Explain symmetry of maximal f-vectors

Outline of the talk

Polytopes defined by oracles: computation & combinatorics

Enumeration of 2-level polytopes

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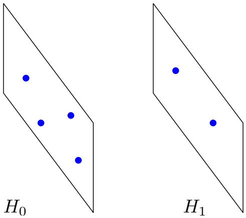
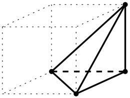
Joint work with:

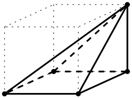
- Adam Bohn (now in Tailand)
- Yuri Faenza (now at EPFL)
- Samuel Fiorini (ULB)
- Marco Macchia (ULB)
- Kanstantsin Pashkovich (now at Waterloo)

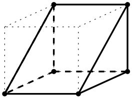


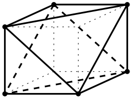
Definition (#1)

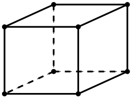
A polytope P is 2-level if $\forall$ facet-defining hyperplane H_0 $\exists$ a parallel hyperplane H_1 such that: $\text{vert}(P) \subseteq H_0 \cup H_1$

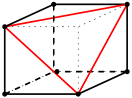

$4, 6, 4$

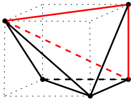

$5, 8, 5$

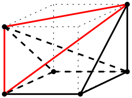

$6, 9, 5$


$6, 12, 8$


$8, 12, 6$


$7, 12, 7$


$5, 9, 6$

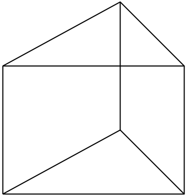

$6, 11, 7$

Definition (#2)

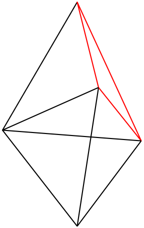
A polytope P is 2-level iff its slack matrix is 0/1 (perhaps after scaling some facets)

Not invariant under polarity:

$P =$



$P^{\Delta} =$



$$S(P) = \begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 \end{pmatrix} \qquad S(P^{\triangle}) = \begin{pmatrix} 0 & 2 & 0 & 0 & 3 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 & 0 \end{pmatrix}$$

Motivations for studying 2-level polytopes

- Algebraic combinatorics / Erhart polynomials (Stanley '80)
- Statistical disclosure elimination (Sullivant '06)
- Centrally symmetric polytopes (Sanyal, Werner, Ziegler '09)
- Theta bodies (Gouveia, Parrilo & Thomas '10)
- Communication complexity (log-rank conjecture)
- Combinatorial optimization (what do 2-level polytopes capture?)

Examples of 2-level polytopes

- Birkhoff polytopes $:=$ convhull of permutation matrices
- Hanner polytopes $:=$ iterated products / free sums of segments
- Stable set polytope $\mathrm{STAB}(G)$ with G perfect
- Hansen polytopes $:=$ twisted prisms over $\mathrm{STAB}(G)$, G perfect
- $\{x \in [0,1]^d \mid Ax = b\}$ where A is totally unimodular and b integer

General properties of 2-level polytopes (2LPs)

- A d-dim 2LP has at most 2^d vertices and facets (GPT '10)
- A polytope is 2-level iff its Theta rank is 1
  (it is a projection of a spectahedron) (GPT '10)
- Every face of a 2LP is a 2LP

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Simplicial cores

In 2-level polytope P, pick

- $d+1$ vertices v_1, …, v_{d+1}
- $d+1$ facets F_1, …, F_{d+1}

such that $\forall i \quad : \quad v_i \notin F_i \quad$ and $\quad v_{i+1}, \ldots, v_{d+1} \in F_i$

Simplicial cores

In 2-level polytope P, pick

- $d+1$ vertices v_1, . . . , v_{d+1}
- $d+1$ facets F_1, . . . , F_{d+1}

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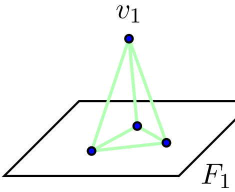

Simplicial cores

In 2-level polytope P, pick

- $d+1$ vertices $v_1, \ldots, v_{d+1}$
- $d+1$ facets $F_1, \ldots, F_{d+1}$

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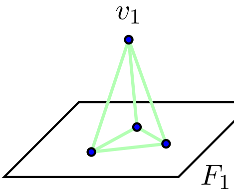


Equivalently, find following submatrix in $S(P)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & 1 & 0 & 0 & \cdots & 0 & 0 \\ * & * & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ * & * & * & * & \cdots & 1 & 0 \\ * & * & * & * & \cdots & * & 1 \end{pmatrix}$$

Embeddings

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & 1 & 0 & 0 & \cdots & 0 & 0 \\ * & * & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ * & * & * & * & \cdots & 1 & 0 \\ * & * & * & * & \cdots & * & 1 \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & M & & \vdots \\ & & & 0 \\ * & \cdots & * & 1 \end{pmatrix}$$

Lemma

For 2L P an x-embedding has 0/1 facets:

$$P = \{x \in \mathbb{R}^d \mid \forall E \in \mathcal{E} : 0 \leqslant \sum_{i \in E} x_i \leqslant 1\}$$

for some $\mathcal{E}$ with subsets of $[d]$ and $\mathrm{vert}(P) \subseteq M^{-1}\{0,1\}^d \subseteq \mathbb{Z}^d$
($\mathcal{E}$ contains all the subsets of $[d]$ if P is the simplex)

Lemma

For 2L P a y-embedding is $P = \mathrm{conv}(X)$ for X $\subseteq \{0,1\}^d$

Remark: The two embeddings linked: $y = Mx \iff x = M^{-1}y$

A proxy for 2LP: closed sets

Definition

$\mathcal{I} := \mathcal{M}^{-1} \cdot \{0,1\}^d$ then $A \subseteq \mathcal{I}$ is closed if $\mathrm{cl}_{\mathcal{I}}(A) = A$.

$$\mathcal{E}(A) := \bigcap_{x \in A} \{E \subseteq [d] \mid 0 \leq x(E) \leq 1\}.$$

$$\mathrm{cl}_{\mathcal{I}}(A) := \{x \in \mathcal{I} \mid 0 \leq x(E) \leq 1 \text{ for every } E \in \mathcal{E}(A)\}.$$

Lemma

If 2L P in x-embedding then the vertex set of P is a closed set wrt $\mathcal{M}^{-1} \cdot \{0,1\}^d$.

The enumeration algorithm

Input: List L_{d-1} of 2L polytopes & simplicial cores

1. **Foreach** $P_1 \in L_{d-1}$ & simplicial core Γ_1: $M_{d-1} := M(\Gamma_1)$
    - 1.1 Complete M_{d-1} to a $(d \times d)$-matrix in the following way:

      $$M_d := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ b_1 & & M_{d-1} & \\ \vdots & & & \\ b_{d-2} & & & \end{pmatrix}$$

      where $b = (b_1, \dots, b_{d-2}) \in \{0,1\}^{d-2}$.
    - 1.2 **Foreach** $b_1, \dots, b_{d-2} \in \{0,1\}$:
        - 1.2.1 Using the Ganter-Reuter algorithm, compute the list $\mathcal{A}$ of closed sets wrt $M_d^{-1} \cdot (\{1\} \times \{0,1\}^{d-1})$
        - 1.2.2 **Foreach** $A \in \mathcal{A}$, let $P := \operatorname{conv}\big((\{0\} \times P_1) \cup A\big)$.
        - 1.2.3 If P is 2-level & not isomorphic to any $P' \in L_d$, add P to L_d.

Experimental results

d	2L	Δ-f	STAB	polar	CS	Birk	0/1	closed sets
3	5	4	4	4	2	4	8	19
4	19	12	11	12	4	11	192	350
5	106	41	33	42	13	33	1,048,576	21239
6	1150	248	148	276	45	129	$\sim 1.8 \cdot 10^{19}$	$1.05 \cdot 10^{8}$
7	-	-	906	-	238	661	-	-
8	-	-	8887	-	-	4530	-	-

Combinatorially equivalent $0/1$ polytopes and 2L polytopes

Δ-f: with on simplicial facet

STAB: stable sets of perfect graphs [Hougardy06]

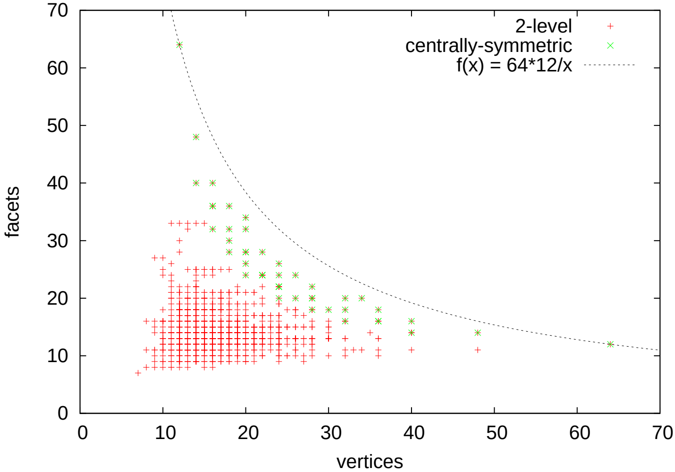
polar: 2-level polytopes whose polar is 2-level

CS: centrally symmetric

Birk: Birkhoff polytope faces [Paffenholz13]

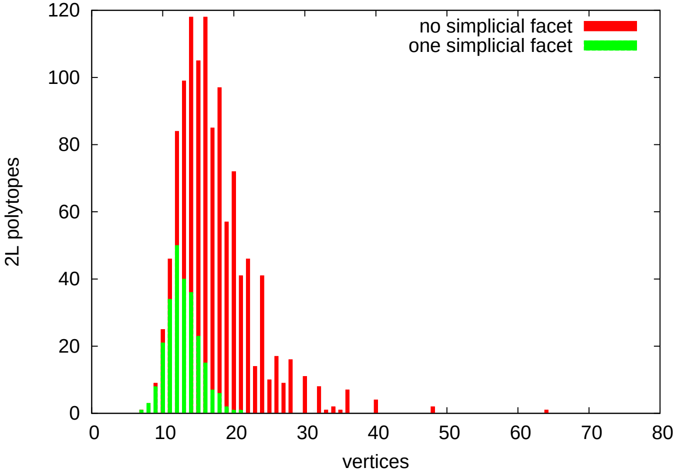
'-': exact numbers are unknown.

Statistics: facets vs vertices



The relation between the number of facets and the number of vertices of 6-dim 2-level polytopes

Statistics: number of 2L



The number of 6-dim 2-level polytopes and the class with the ones with a simplicial facet as a function of the number of vertices.

Open questions

- Output-sensitive enumeration algorithm for 2L polytopes (Hint: better proxy)
- $g(d) := \#(d$-dimensional 2L polytopes, up to isomorphism)
  Is $g(d) = 2^{\mathrm{poly}(d)}$?
- Known: $g(d) \geqslant 2^{\Omega(d^2)}$ (e.g., $\mathrm{STAB}(G)$ with G bipartite)

Open questions

- Output-sensitive enumeration algorithm for 2L polytopes
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$\mathrm{xc}(P)$ = extension complexity = min # facets of a lift of P

- $f(d) := \max\{\mathrm{xc}(P) \mid P$ is d-dimensional 2L polytope$\}$
  Is $f(d) = 2^{\mathrm{polylog}(d)}$? ("log-rank conj. for slack matrices")
- $\mathrm{xc}(\mathrm{STAB}(G)) \leqslant 2^{O(\log^2 n)}$ for n-vertex perfect G (Yannakakis'91)
- $f(d) \leqslant 2^{\tilde{O}(\sqrt{d})}$ for d-dimensional 2LP (Lovett '14)
- $g(d) \leqslant 2^{O[\mathrm{poly}(d) f^2(d)]}$ (Rothvoss '11)

Thank you!