Exact and approximate algorithms for resultant polytopes

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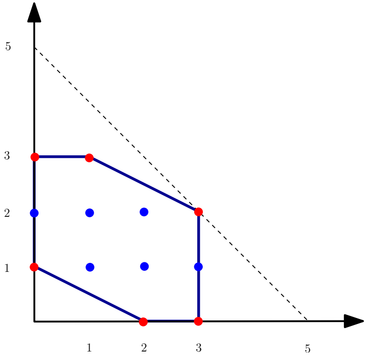
Polynomials and Newton polytopes

- The support of a polynomial f is the set of exponents of its monomials with non-zero coefficient.
- The Newton polytope of f is the convex hull of its support.

$f(x, y) =$
$8y + xy - 24y^2 - 16x^2 + 220x^2y - 34xy^2 - 84x^3y + 6x^2y^2 - 8xy^3 + 8x^3y^2 + 8x^3 + 18y^3$



Polynomial systems

We study polynomials that expresses the solvability of polynomial systems.

Given a system of $n+1$ linear polynomials $f_0, f_1, \ldots, f_n$, on n variables the determinant is a polynomial on the coefficients which is zero iff the system has a common solution.

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$$f_0 = ax + by + c = 0$$
$$f_1 = dx + ey + f = 0$$
$$f_2 = gx + hy + i = 0$$

$$\begin{vmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{vmatrix}$$

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Given a system of $n+1$ linear polynomials $f_0, f_1, \ldots, f_n$, on n variables the determinant is a polynomial on the coefficients which is zero iff the system has a common solution.

$$f_0 = 4x + y + 2 = 0$$
$$f_1 = x + 2y + 1 = 0$$
$$f_2 = x + y + 8 = 0$$

$$\begin{vmatrix} 4 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 8 \end{vmatrix} = 51$$

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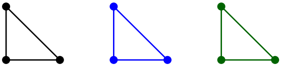
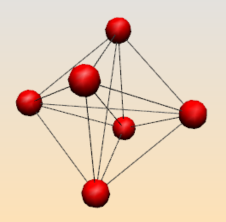


Newton polytope of determinant (Birkhoff polytope)



$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Polynomial systems

Given a system of $n+1$ ~~linear~~ general polynomials $f_0, f_1, \ldots, f_n$, on n variables the ~~determinant~~ resultant is a polynomial on the coefficients which is zero iff the system has a common solution.

$f_0 = 4xy^2 + x^4y + 2 = 0$
$f_1 = x + 2y = 0$
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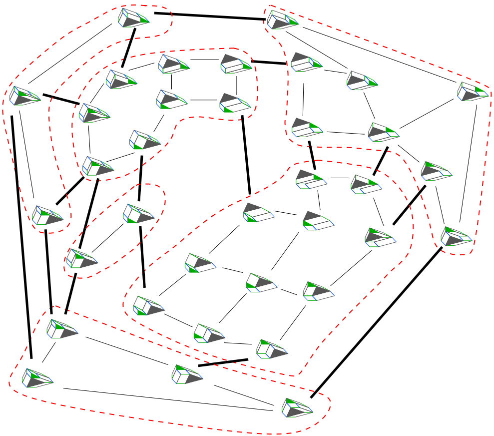
The idea of the algorithm

The input supports define a pointset $\mathcal{A} \in \mathbb{Z}^{2n}$

Naive method: compute the secondary polytope $\Sigma(\mathcal{A})$ to compute Π



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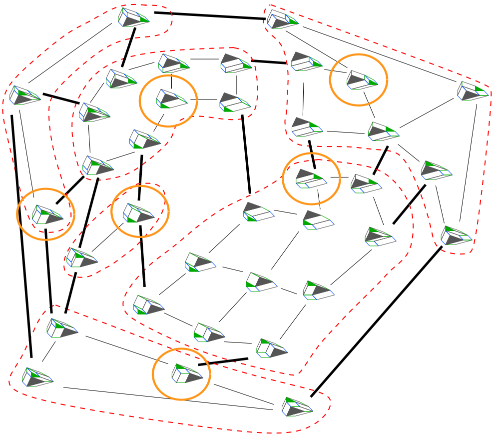
Idea: incrementally construct Π using an **oracle** that given a direction produces vertices of Π

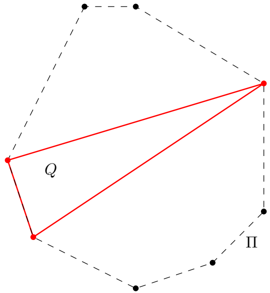


Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = \Pi$

1. initialization step



initialization:

- $Q \subset \Pi$
- dim(Q)=dim(Π)

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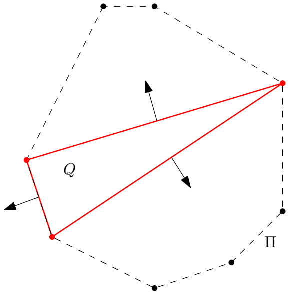

2 kinds of hyperplanes of Q_H:

- legal if it supports facet $\subset \mathit{\Pi}$
- illegal otherwise

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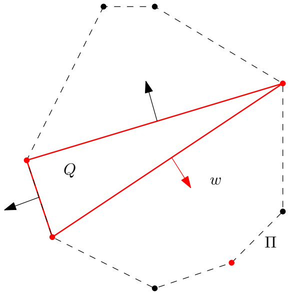

Extending an illegal facet

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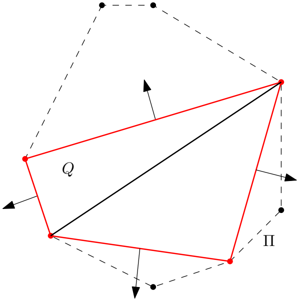

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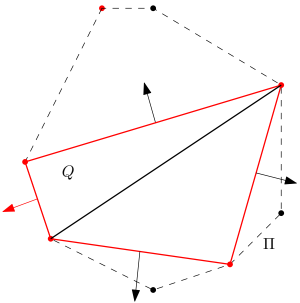

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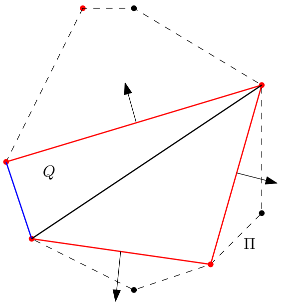

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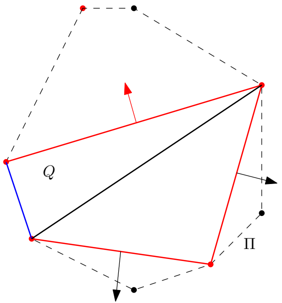

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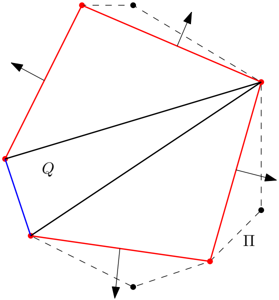

At any step, Q is an inner approximation ...

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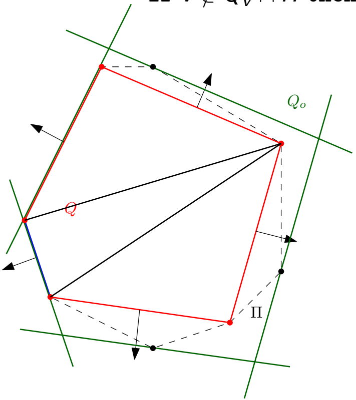

At any step, Q is an inner approximation ... from which we can compute an outer approximation Q_o.

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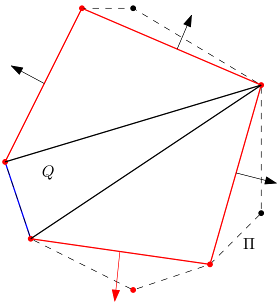

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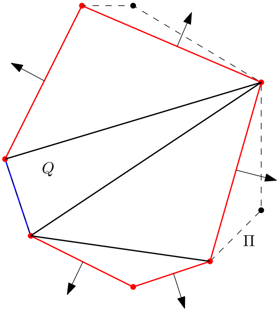

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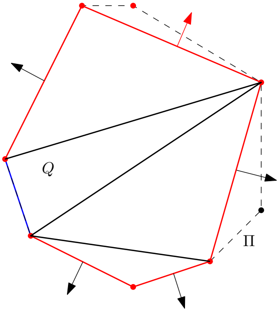

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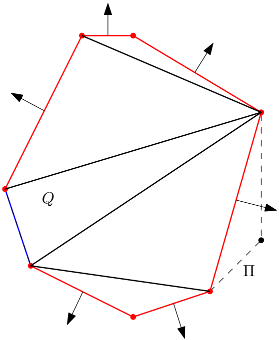

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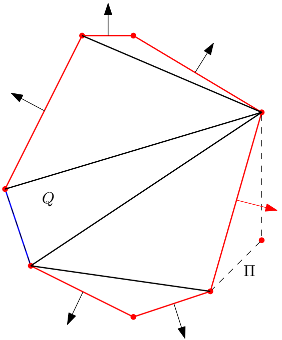

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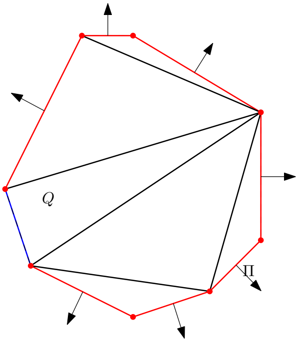

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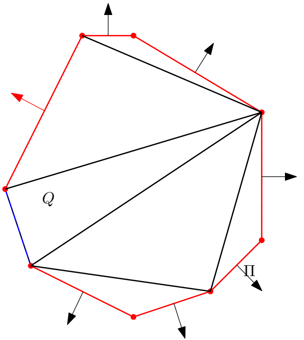

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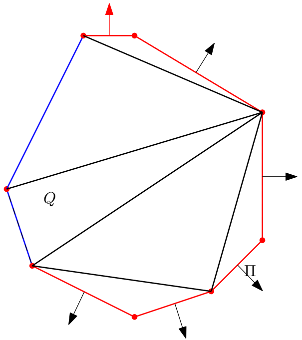

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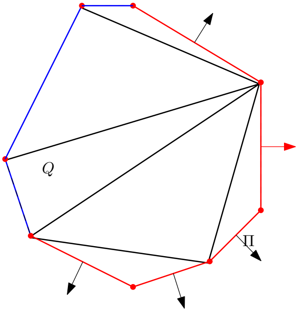

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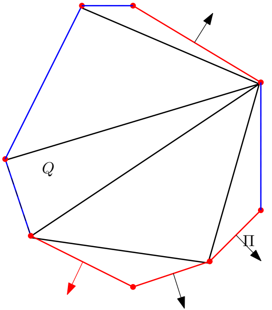

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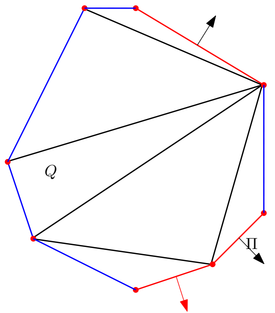

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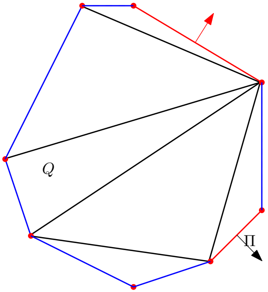

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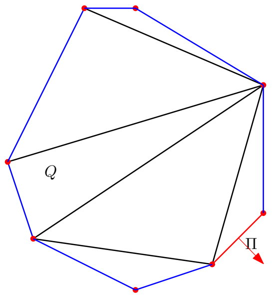

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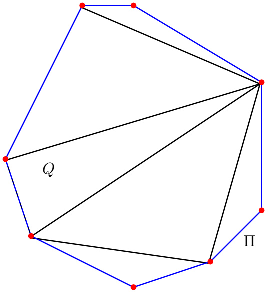


Complexity

Theorem

We compute the Vertex- and Halfspace-representations of Π, as well as a triangulation T of Π, in

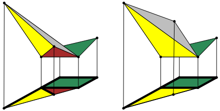
$$O^*(\, m^5\, |vtx(\Pi)| \cdot |T|^2),$$

where $m = \dim \Pi$, and $|T|$ the number of full-dim faces of T.

Elements of proof

- Most computation is done in dimension $\leq m$.
- At most $\leq \mathrm{vtx}(\Pi) + \mathrm{fct}(\Pi)$ oracle calls
- Beneath-Beyond algorithm for converting V-rep. to H-rep. (bottleneck)

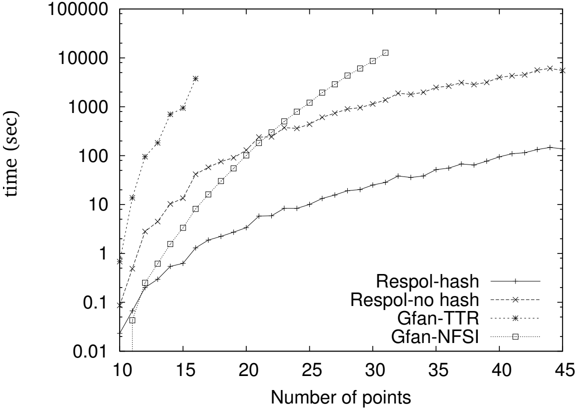
ResPol Implementation



Tools

C++, CGAL, `triangulation` [Boissonnat,Devillers,Hornus], `extreme_points_d` [Gärtner]

Experiments ($dim(\Pi) = 4$)



Future work

- approximate resultant polytopes ($dim(\Pi) \geq 7$)
- preliminary results:

input	m	3	3	4	4	5	5
	$\lvert\mathcal{A}\rvert$	200	490	20	30	17	20
exact	#vtx(Π)	98	133	416	1296	1674	5093
	time	2.03	5.87	3.72	25.97	51.54	239.96
approx.	#vtx(Q_{in})	15	11	63	121	–	–
	vol(Q_{in})/vol(Π)	0.96	0.95	0.93	0.94	–	–
	vol(Q_{out})/vol(Π)	1.02	1.03	1.04	1.03	–	–
	time	0.15	0.22	0.37	1.42	> 10hr	> 10hr

- approximate volume computation [Lovász-Vempala06]

References

The code

- `http://respol.sourceforge.net`

The full version of the paper

- `http://arxiv.org/abs/1108.5985v2`

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The code

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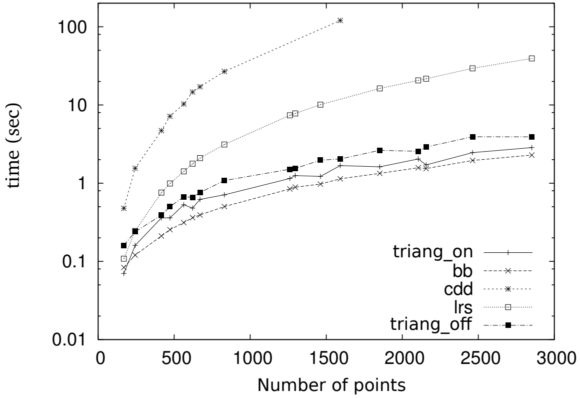
Thank You !

Convex hull implementations

- From V- to H-rep. of Π.
- `triangulation (on/off-line)`, polymake `beneath-beyond`, `cdd`, `lrs`



$dim(\Pi) = 4$