Enumeration of 2-level polytopes

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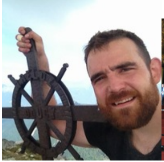
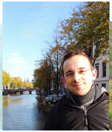
Enumeration of 2-level polytopes *(to appear in ESA'15)*

Joint work with:

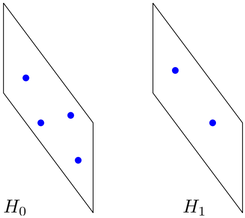
- Adam Bohn (now in Tailand)
- Yuri Faenza (now at EPFL)
- Samuel Fiorini (ULB)
- Marco Macchia (ULB)
- Kanstantsin Pashkovich (now at Waterloo)




Definition (#1)

A polytope P is 2-level if for EVERY facet-defining hyperplane H_0 $\exists$ a parallel hyperplane H_1 such that: vert$(P) \subseteq H_0 \cup H_1$



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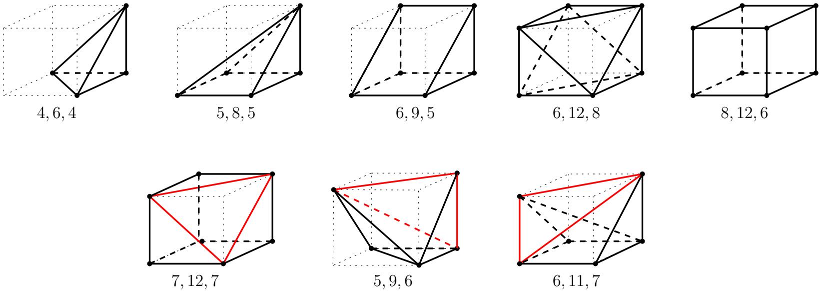
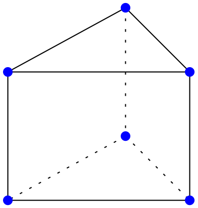


Rem: All 2-level can be embedded as $0/1$ polytopes

Definition

The slack matrix of a polytope $P \subseteq \mathbb{R}^d$ with m facets $F_1, \ldots, F_m$ and n vertices $v_1, \ldots, v_n$ is the $m \times n$ non-negative matrix $S(P)$ s.t. $S_{ij} = g_i(v_j)$ where g_i is the i-th facet defining inequality

$P =$ 

$$S(P) = \begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 \end{pmatrix}$$

Definition (#2)

A polytope P is 2-level iff its slack matrix is 0/1 (perhaps after scaling some facets)

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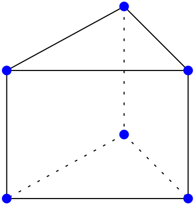
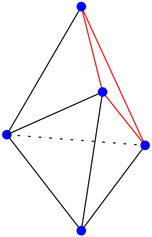
Not invariant under polarity:

$P =$ 

$P^{\Delta} =$ 

$$S(P) = \begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 \end{pmatrix}$$

$$S(P^{\triangle}) = \begin{pmatrix} 0 & 2 & 0 & 0 & 3 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 & 0 \end{pmatrix}$$

Examples of 2-level polytopes

- Birkhoff polytopes $:=$ convhull of permutation matrices
- Order polytopes $:=$ convhull of characteristic vectors of all partial orders of a set
- Hanner polytopes $:=$ iterated products / free sums of segments
- Stable set polytope $\mathrm{STAB}(G)$ with G perfect graph
- Hansen polytopes $:=$ twisted prisms over $\mathrm{STAB}(G)$, G perfect
- $\{x \in [0,1]^d \mid Ax = b\}$ where A is totally unimodular and b integer

General properties of 2-level polytopes (2LPs)

- A d-dim 2LP has at most 2^d vertices and facets [GPT '10]
- Every face of a 2LP is a 2LP

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- A d-dim 2LP has at most 2^d vertices and facets [GPT '10]
- Every face of a 2LP is a 2LP
- The combinatorial type of a 2LP determines its affine type
- A 2LP has a simple vertex iff it is $\simeq \mathrm{STAB}(G)$, G perfect

Motivations for studying 2-level polytopes

- Algebraic combinatorics / Erhart polynomials (Stanley '80)
- Statistical disclosure elimination (Sullivant '06)
- Centrally symmetric polytopes (Sanyal, Werner, Ziegler '09)
- Theta bodies (Gouveia, Parrilo & Thomas '10)
- Communication complexity
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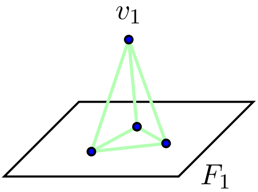
Today: enumerate 2L polytopes in fixed dimension

Simplicial cores

In 2-level polytope P, pick

- $d+1$ vertices $v_1, \ldots, v_{d+1}$
- $d+1$ facets $F_1, \ldots, F_{d+1}$

such that $\forall i \quad : \quad v_i \notin F_i$ and $v_{i+1}, \ldots, v_{d+1} \in F_i$



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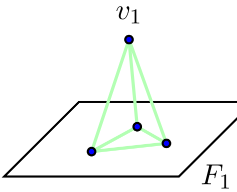


Equivalently, find following submatrix in $S(P)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & 1 & 0 & 0 & \cdots & 0 & 0 \\ * & * & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ * & * & * & * & \cdots & 1 & 0 \\ * & * & * & * & \cdots & * & 1 \end{pmatrix}$$

H-embedding or 0/1 facets

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & 1 & 0 & 0 & \cdots & 0 & 0 \\ * & * & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ * & * & * & * & \cdots & 1 & 0 \\ * & * & * & * & \cdots & * & 1 \end{pmatrix} = M$$

Lemma

For 2L P an H-embedding has 0/1 facets:

$$P = \{x \in \mathbb{R}^d \mid \forall E \in \mathcal{E} : 0 \leqslant \sum_{i \in E} x_i \leqslant 1\}$$

for some $\mathcal{E}$ with subsets of $[d]$ and $\mathrm{vert}(P) \subseteq M^{-1}\{0,1\}^d \subseteq \mathbb{Z}^d$

V-embedding or 0/1 vertices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & 1 & 0 & 0 & \cdots & 0 & 0 \\ * & * & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ * & * & * & * & \cdots & 1 & 0 \\ * & * & * & * & \cdots & * & 1 \end{pmatrix} = M$$

Lemma

For 2L P a V-embedding is $P = \mathrm{conv}(X)$ for $X \subseteq \{0,1\}^d$

Remark: The two embeddings linked:

$$y = Mx \iff x = M^{-1}y$$

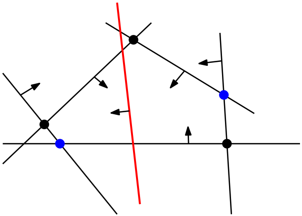
$y \in \{0,1\}^d$, $x \in \mathbb{Z}^d$

A proxy for 2LP: closed sets

$\mathcal{I} := \mathcal{M}^{-1} \cdot \{0,1\}^d$ and $A \subseteq \mathcal{I}$

$\mathrm{cl}_{\mathcal{I}}(A) := \{x \in \mathcal{I} \mid x$ validates the valid inequalities for $A\}$

Def. A is closed if $\mathrm{cl}_{\mathcal{I}}(A) = A$.



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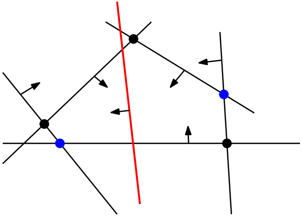

Lemma

If 2L P in H-embedding then the vertex set of P is a closed set w.r.t. $\mathcal{M}^{-1} \cdot \{0,1\}^d$.

The enumeration algorithm

Input: List L_{d-1} of $d-1$-dim 2L polytopes & simplicial cores

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1. **Foreach** $P_1 \in L_{d-1}$ & simplicial core Γ_1: $M_{d-1} := M(\Gamma_1)$
   1.1 Complete M_{d-1} to a $(d \times d)$-matrix in the following way:

$$M_d := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ b_1 & & & \\ \vdots & & M_{d-1} & \\ b_{d-2} & & & \end{pmatrix}$$ where $(b_1, \ldots, b_{d-2}) \in \{0,1\}^{d-2}$.

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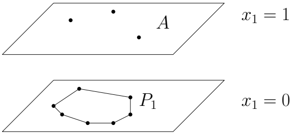
     - 1.2.2 **Foreach** $A \in \mathcal{A}$, let $P := \operatorname{conv}\big((\{0\} \times P_1) \cup A\big)$.
     - 1.2.3 If P is 2-level & not isomorphic to any $P' \in L_d$, add P to L_d.



Experimental results

d	2L	Δ-f	STAB	polar	CS	Birk	0/1	closed sets
3	5	4	4	4	2	4	8	19
4	19	12	11	12	4	11	192	277
5	106	41	33	42	13	33	$\sim 10^6$	10963
6	1150	248	148	276	45	129	$\sim 1.8 \cdot 10^{19}$	$\sim 1.9 \cdot 10^6$
7	-	-	-	-	238	661	-	-

Combinatorially equivalent $0/1$ polytopes and 2L polytopes

Δ-f: with on simplicial facet

STAB: stable sets polytopes of perfect graphs

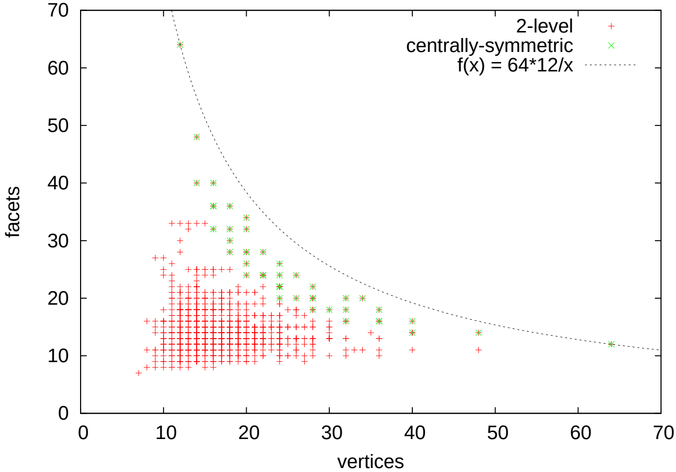
polar: 2-level polytopes whose polar is 2-level

CS: centrally symmetric

Birk: Birkhoff polytope faces [Paffenholz13]

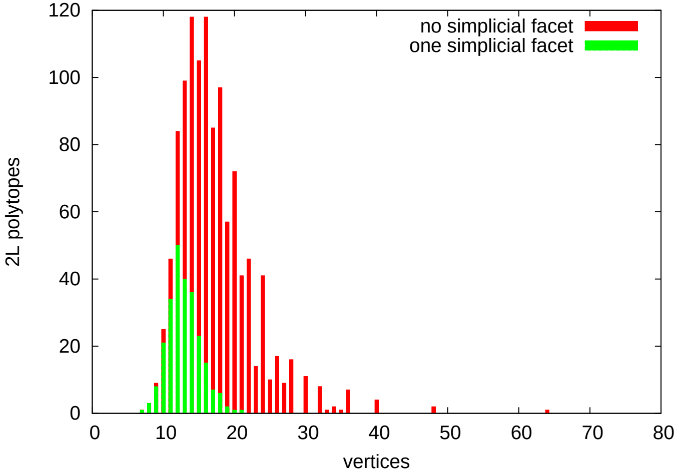
'-': exact numbers are unknown.

Statistics: facets vs vertices



Number of facets and the number of vertices of 6-dim 2-level polytopes and the $d * 2^{d+1}$ conjecture

Statistics: number of 2L



The number of 6-dim 2-level polytopes and the class with the ones with a simplicial facet as a function of the number of vertices.

Open questions

- Output-sensitive enumeration algorithm for 2L polytopes (Hint: better proxy)
- $g(d) := \#(d$-dimensional 2L polytopes, up to isomorphism)
  Is $g(d) = 2^{\mathrm{poly}(d)}$?
- Known: $g(d) \geqslant 2^{\Omega(d^2)}$ (e.g., $\mathrm{STAB}(G)$ with G bipartite)

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THANK YOU!